Linearised Higher Variational Equations

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Abstract

This work explores the tensor and combinatorial constructs underlying the linearised higher-order variational equations LVE_{ϕ}^{k} of a generic autonomous system along a particular solution $\boldsymbol{\phi}$. The main result of this paper is a compact yet explicit and computationally amenable form for said variational systems and their monodromy matrices. Alternatively, the same methods are useful to retrieve, and sometimes simplify, systems satisfied by the coefficients of the Taylor expansion of a formal first integral for a given dynamical system. This is done in preparation for further results within Ziglin-Morales-Ramis theory, specifically those of a constructive nature.

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1 Motivation and first definitions

1.1 Introduction

Integrability, an informal word for reasonably simple solvability, is an important problem in Dynamical Systems. Its opposite phenomenon, and specifically low predictability with respect to time, is usually summarised under the term *chaos*. If the system is Hamiltonian, as are most problems in Mechanics, the "chaos vs solvability" disjunctive is doubly advantageous. On one hand, it is amenable to the techniques of Symplectic Geometry. On the other, theory and empirics yield a specific, thus observable integrability condition: the existence of a precise amount of conserved quantities.

The introduction of the algebraic approach by Ziglin, Morales-Ruiz and Ramis produced hallmark contributions to the study of the integrability of Hamiltonian systems [6, 22, 23, 30], essentially couched on a study of the invariants of a given matrix group, associated to a linear system: the *first-order variational equations* introduced in 1.2.

A second step forward was carried out by Morales-Ruiz, Ramis and Simó ([24]) in order to extend the preceding theoretical framework to the Galois groups of the (linearised) higher-order variational equations along a particular solution.

The second step described in the previous paragraph is the driving force behind this paper. A constructive version of the Morales-Ramis-Simó theorem was already started by Aparicio-Monforte and Weil in [2] and tangentially tackled from another viewpoint in [5] (see Section 5) and the present work aims at expanding this effort by offering a closed-form expression for the linearised higher variationals. May the reader bear in mind that at no stage in the results of this paper from Section 2 onwards is the system required to be Hamiltonian.

1.2 Dynamical systems and variational equations

In accordance with results succinctly described in 1.3 and thereafter, we need to observe the following convention outside of Sections 2 and 3: dependent and independent variables for all dynamical systems will be allowed to be *complex*. Any open set $T \subseteq \mathbb{P}^1_{\mathbb{C}}$ is an admissible domain for the time variable, embedded into the Riemann sphere to include $t = \infty$ as a valid singularity.

Consider an autonomous dynamical system:

$$\dot{\boldsymbol{z}} = X(\boldsymbol{z}), \qquad \text{where } X : U \subseteq \mathbb{C}^n \to \mathbb{C}^n. \tag{DS}$$

Assume X is holomorphic. Basic mathematical objects are defined analogously to their real-valued counterparts: conserved quantities and foliations of solution curves.

Definition 1.1. (DS) *given, assume*

*a) A **first integral** of* (DS) *is a function $F : V \supset U \to \mathbb{C}$ constant along every solution of* (DS)*. Equivalently, such that $D_X F = 0$, where $D_X := \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$.*

b) For every $\boldsymbol{z} \in U$, let $\varphi(t, \boldsymbol{z})$ be the unique solution of (DS) *such that $\varphi(0, \boldsymbol{z}) = \boldsymbol{z}$, defined on a maximal open set $I(\boldsymbol{z})$. Function $\varphi : I(\boldsymbol{z}) \times U \to \mathbb{C}^n$ thus defined is $\mathcal{C}^1$ and whenever $\boldsymbol{y} = \varphi(\tau, \boldsymbol{x})$ for some $\tau \in I(\boldsymbol{z})$, translation $I(\boldsymbol{y}) = I(\boldsymbol{z}) - \tau$ holds and $\varphi(t, \boldsymbol{y}) = \varphi(\tau + t, \boldsymbol{z})$, for every $t \in I(\boldsymbol{y})$. φ is called the **flow** of* (DS)*.*

Clarifying preliminary comments are in order whenever a particular solution $\boldsymbol{\phi}(t)$ is considered:

a) trivially, the partial derivatives $\frac{\partial^k}{\partial \boldsymbol{z}^k}\varphi(t, \boldsymbol{\phi})$ of the flow are multilinear forms of increasing order (or, alternatively, multidimensional matrices, see e.g. [17]) and may also be characterised as the blocks appearing in the Taylor expression of the flow along $\boldsymbol{\phi}$, minus the

factorial denominators:

$$\varphi(t,\boldsymbol{z}) = \varphi(t,\boldsymbol{\phi}) + \frac{\partial\varphi(t,\boldsymbol{\phi})}{\partial\boldsymbol{z}}\{\boldsymbol{z}-\boldsymbol{\phi}\} + \frac{1}{2!}\frac{\partial^2\varphi(t,\boldsymbol{\phi})}{\partial\boldsymbol{z}^2}\{\boldsymbol{z}-\boldsymbol{\phi}\}^2 + \frac{1}{3!}\frac{\partial^3\varphi(t,\boldsymbol{\phi})}{\partial\boldsymbol{z}^3}\{\boldsymbol{z}-\boldsymbol{\phi}\}^3 + \ldots, \quad (1)$$

bracket notation summarising multilinear forms.

b) each of these partial derivatives $\frac{\partial^k}{\partial\boldsymbol{z}^k}\varphi(t,\boldsymbol{\phi})$, inverse factorial unaccounted for, may also be characterised as satisfying an echeloned set of differential systems, depending on the previous $k-1$ partial derivatives and customarily called **variational equations or systems**. They are explicitly called **higher-order** whenever $k \geqslant 2$.

c) the variational system corresponding to $k=1$ is linear:

$$\dot{Y}_1 = A_1 Y_1, \qquad A_1(t) := \left.\frac{\partial X}{\partial\boldsymbol{z}}\right|_{\boldsymbol{z}=\boldsymbol{\phi}(t)} \in \mathrm{Mat}_n(K) \qquad (\mathrm{VE}_\phi)$$

its principal fundamental matrix being the linear part of the flow along ϕ, and $K = \mathbb{C}(\phi)$ being the smallest differential field containing $\mathbb{C}(t)$ and the solution.

d) For $k \geqslant 2$, however, the system is not linear, yet a linearised version may be found. *The aim of the present paper is to do so with explicit formulae.*

1.3 Morales-Ramis-Ziglin theory and extensions

Heuristics of all non-integrability results within the Ziglin-Morales-Ramis-Simó theoretical framework are firmly rooted in the following principle, expected to affect a widespread class of systems:

> *If we assume general system (DS) "integrable" in some reasonable sense, then for every particular solution $\boldsymbol{\phi}(t)$ of (DS) the differential system satisfied by each of the partial derivatives of the flow at $\boldsymbol{\phi}(t)$ must be also integrable in an accordingly reasonable sense.*

Any attempt at ad-hoc formulations of this heuristic principle for (DS) has an asset and a drawback:

- there *is* a valid integrability axiom for linear systems (and thus, for (VE_ϕ)): that the identity component of an algebraic group attached to them, named the *differential Galois group* [22, 28], be *solvable*;
- still, in order to transform this principle into a true conjecture it is necessary to clarify a notion of "integrability " for (DS).

The latter item is cleared in the Hamiltonian case by the *Liouville-Arnold Theorem* establishing a sufficient condition for a system to admit, at least locally, a new set of variables rendering it integrable by quadratures. Said condition is the hypothesis on H in the following:

Theorem 1.2 (Morales-Ruiz, Ramis, 2001)**.** *Let X_H be an n-degree-of-freedom Hamiltonian system having n independent first integrals in pairwise involution, defined on a neighborhood of an integral curve ϕ. Then, the identity component of the Galois group of the variational equations of H along ϕ is an abelian group.* □

See [23, Cor. 8] or [22, Th. 4.1] for a precise statement and a proof.

Theorem 1.3 (Morales-Ruiz, Ramis, Simó, 2005, [24, Th. 5])**.** *Let H be as in the previous theorem. Let G_k be the differential Galois group of the k-th variational equations, $k \geqslant 1$, and $G := \varprojlim G_k$ the formal differential Galois group (inverse limit of the groups) of X_H along ϕ. Then, the identity components of the Galois groups G_k and G are abelian.* □

Theorem 1.3 makes use of variational equations defined as in Section 1.2 and the language of jets, after proving said non-linear equations equivalent, in practicality, to *any* consistent linearised completion. Efforts towards a constructive version of this main Theorem, as well as the line of study described in Section 5, are hampered by a lack of consensus on the explicit block structure of said linearised completion. The present work, summarised in its main result (Proposition 4.5) aims at contributing to fill in this gap. Hence, outcomes will be purely restricted to symbolic calculus and do not constitute new results in the theoretical framework summarised above.

Notation 1.4 (Multi-indices and lexicographic order)**.** Part of the conventions listed below were already introduced in [5].

1. The *modulus* $i = |\mathbf{i}|$ of a multi-index $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{Z}^n$ is the sum of its entries. Multi-index addition and subtraction are defined entrywise as usual.

2. We use the *standard lexicographic order*: $(i_1, \ldots, i_n) < (j_1, \ldots, j_n)$ means $i_1 = j_1, \ldots, i_{k-1} = j_{k-1}$ and $i_k < j_k$ for some $k \geqslant 1$.

3. Let $F : U \subset \mathbb{C}^n \to \mathbb{C}$ be a complex analytic function over the open set U. We define the *lexicographically sifted differential of F of order m* as the row vector

$$F^{(m)}(x) := \operatorname{lex}\left(\frac{\partial^m F}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}(x)\right),$$

where $i_1 + \ldots + i_n = m$ and entries are ordered as per $<_{\mathrm{lex}}$ on multi-indices.

4. We define

$$d_{n,k} := \binom{n+k-1}{n-1}, \qquad D_{n,k} := d_{n,1} + d_{n,2} + \cdots + d_{n,k}. \tag{2}$$

It is easy to check that the set of possible k-ples of integers in $\{1, \ldots, n\}$, or alternatively the number of homogeneous monomials of total degree n in k variables, has $d_{n,k}$ elements. Quantity $D_{n,k}$ will become useful in Section 4.1 when LVE_ϕ^k is introduced.

Notation 1.5 (Multi-index binomials and multinomials)**.** Given integers $k_1, \ldots, k_n \geqslant 0$, we define the usual multinomial coefficient as

$$\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} := \binom{k_1 + \cdots + k_n}{\mathbf{k}} := \frac{(k_1 + \cdots + k_n)!}{k_1! k_2! \cdots k_n!}. \tag{3}$$

For a multi-index $\mathbf{k} \in \mathbb{Z}^n_{\geqslant 0}$, define $\mathbf{k}! := k_1! \cdots k_n!$. For any two such $\mathbf{k}, \mathbf{j}$, we define

$$\binom{\mathbf{k}}{\mathbf{p}} := \frac{k_1! k_2! \cdots k_n!}{p_1! p_2! \cdots p_n! \,(k_1 - p_1)!\,(k_2 - p_2)! \cdots (k_n - p_n)!} = \binom{k_1}{p_1}\binom{k_2}{p_2} \cdots \binom{k_n}{p_n}, \tag{4}$$

and the multi-index counterpart to (3),

$$\binom{\mathbf{k}_1 + \cdots + \mathbf{k}_m}{\mathbf{k}_1, \ldots, \mathbf{k}_m} := \frac{(\mathbf{k}_1 + \cdots + \mathbf{k}_n)!}{\mathbf{k}_1! \mathbf{k}_2! \cdots \mathbf{k}_n!}. \tag{5}$$

2 Symmetric products and powers of finite matrices

2.1 Definition and properties

Let K be a field and V a K-vector space. Let us first recount the definition and the requisite existence and uniqueness results for the symmetric power of V. See [13, 14, 18] for details.

Definition 2.1. *An r^{th} **symmetric tensor power** of V is a vector space S, together with a symmetric multilinear map $\varphi : V^r := V \times \overset{r}{\cdots} \times V \to S$ satisfying the following* universal property*: for every vector space W and every symmetric multilinear map $f : V^r \to W$ there is a unique induced linear map $f_\odot : S \to W$ such that the following diagram commutes:*

$$\begin{tikzcd} V \times V \times \overset{r}{\cdots} \times V \arrow[r, "f"] \arrow[d, "\varphi"'] & W \\ S \arrow[ur, dashed, "f_\odot"'] & \end{tikzcd}$$

In other words, isomorphism $\operatorname{Hom}(S, W) \cong S(V^n, W)$ *holds between the vector space of linear maps $S \to W$ and the vector space of symmetric multilinear maps $V^n \to W$.*

Lemma 2.2. *For any two symmetric r^{th} powers (S_1, φ_1) and (S_2, φ_2) of V, an isomorphism $\psi : S_1 \to S_2$ exists such that $\varphi_2 = \psi \circ \varphi_1$.*

Proposition 2.3. *Given any vector space V and any $r \in \mathbb{N}$,*

a) a symmetric r^{th} power (S, φ) exists for V. We denote:

- $\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_r := \varphi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$ *for every* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in V$,
- $\operatorname{Sym}^r V$, $V \odot \overset{r}{\cdots} \odot V$ *or* $\bigodot^r V$ *in place of* S,
- $\boldsymbol{v}^{\odot k} := \boldsymbol{v} \odot \overset{k}{\cdots} \odot \boldsymbol{v}$ *for any vector* $\boldsymbol{v} \in V$, *and*
- $\boldsymbol{v}^{\odot \mathbf{p}} := \boldsymbol{v}_1^{\odot p_1} \odot \cdots \odot \boldsymbol{v}_n^{\odot p_n}$, *for any set of vectors* $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ *and multi-index* $\mathbf{p} \in \mathbb{Z}^n_{\geqslant 0}$.

Conventions $\operatorname{Sym}^1 V = V$ *and* $\operatorname{Sym}^0 V = K$ *arise naturally.*

b) Furthermore, $\{\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_r : \boldsymbol{v}_1, \cdots, \boldsymbol{v}_r \in V\}$ is a system of generators of $\operatorname{Sym}^r V$.

c) For any vector space W and multilinear map $f : V^r \to W$, the linear map $f_\odot$ induced by the universal property is defined on the set of generators of $\operatorname{Sym}^r V$ *as*

$$f_\odot(\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_r) = f(\boldsymbol{v}_1, \cdots, \boldsymbol{v}_r).$$

d) If $\dim V = n < \infty$ *then every basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of V induces a basis for* $\operatorname{Sym}^r V$:

$$\{(\boldsymbol{e}_1 \odot \overset{r_1}{\cdots} \odot \boldsymbol{e}_1) \odot (\boldsymbol{e}_2 \odot \overset{r_2}{\cdots} \odot \boldsymbol{e}_2) \odot \cdots \odot (\boldsymbol{e}_n \odot \overset{r_n}{\cdots} \odot \boldsymbol{e}_n) : r_i \geqslant 0,\ |\mathbf{r}| = r\}; \qquad (6)$$

hence, $\dim \operatorname{Sym}^r V = d_{n,r}$.

In particular, symmetric products of vectors operate exactly like products of homogeneous polynomials in n variables.

Remark 2.4. Sym^r may also be defined explicitly in terms of the tensor power $\bigotimes^r$, delegating observation of a universal property on the latter and then taking quotients $\operatorname{Sym}^r V = \bigotimes^r V / \sim$ modulo the equivalence relation $\boldsymbol{v}_1 \otimes \cdots \otimes \boldsymbol{v}_r \sim \boldsymbol{v}_{\sigma(1)} \otimes \cdots \otimes \boldsymbol{v}_{\sigma(r)}$ for every $\sigma \in \mathfrak{S}_r$, thus equating, via isomorphism, $\operatorname{Sym}^r V$ to the subspace $\langle \{\boldsymbol{e}_{i_1} \otimes \boldsymbol{e}_{i_2} \otimes \cdots \otimes \boldsymbol{e}_{i_r} : 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r \leqslant n\} \rangle$ of $V^{\otimes r}$ for any basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of V.

Given any K-vector space W and two linear maps $f, g : V \to W$, define

$$h : V \times V \to \mathrm{Sym}^2 W, \qquad h(\boldsymbol{v}_1, \boldsymbol{v}_2) := \frac{1}{2}\left[f(\boldsymbol{v}_1) \odot g(\boldsymbol{v}_2) + f(\boldsymbol{v}_2) \odot g(\boldsymbol{v}_1)\right]. \tag{7}$$

Immediately bilinear and symmetric, it is granted a unique linear $h_\odot : \mathrm{Sym}^2 V \to \mathrm{Sym}^2 W$, obviously defined $h_\odot(\boldsymbol{v}_1 \odot \boldsymbol{v}_2) := h(\boldsymbol{v}_1, \boldsymbol{v}_2)$, by the universal property. Let us write $f \odot g := h_\odot$. It is easy to check that $f \odot g = g \odot f$ and, given linear maps $f_1, g_1 : W \to W_1$,

$$(f_1 \circ f) \odot (g_1 \circ g) = (f_1 \odot g_1) \circ (f \odot g).$$

A similar construction applies to the symmetric product of $m \geqslant 3$ linear maps $f_i : V \to W$:

$$f_1 \odot \cdots \odot f_m : \mathrm{Sym}^m V \to \mathrm{Sym}^m W, \quad \boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_m \mapsto \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} f_1\left(\boldsymbol{v}_{\sigma(1)}\right) \odot \cdots \odot f_m\left(\boldsymbol{v}_{\sigma(m)}\right). \tag{8}$$

Let us generalise the above symmetric product into one involving any two linear maps

$$f : \mathrm{Sym}^{j_1} V \to \mathrm{Sym}^{i_1} W, \quad g : \mathrm{Sym}^{j_2} V \to \mathrm{Sym}^{i_2} W, \qquad j_1, j_2, i_1, i_2 \geqslant 0.$$

Assume V and W finite-dimensional, V having basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. We will use notation in Proposition 2.3. Defining the bilinear map

$$\varphi(\boldsymbol{u}_1, \boldsymbol{u}_2) := \boldsymbol{u}_1 \odot \boldsymbol{u}_2, \qquad \boldsymbol{u}_i \in \mathrm{Sym}^{j_i} V, \quad i = 1, 2, \tag{9}$$

we look forward to building a new symmetric bilinear function h in terms of f and g generalising (7), and proving there is a unique linear $h_\odot$ completing the diagram

$$\begin{tikzcd} \mathrm{Sym}^{j_1} V \times \mathrm{Sym}^{j_2} V \arrow[r, "h"] \arrow[d, "\varphi"'] & \mathrm{Sym}^{i_1+i_2} W \\ \mathrm{Sym}^{j_1+j_2} V \arrow[ur, dashed, "h_\odot"'] & \end{tikzcd} \tag{10}$$

We want h to be a symmetric, bilinear map depending on f and g and yielding coefficient 1 for all-round repeated vectors as in (7). Symmetric, multilinear $\tilde{h} : V^{\times j_1 + j_2} \to \mathrm{Sym}^{i_1+i_2} W$ is easier to define, generalising (7): for any $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{j_1+j_2} \in V$,

$$\tilde{h}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{j_1+j_2}) := \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{\sigma \in S_{j_1,j_2}} f\left(\boldsymbol{u}_{\sigma(1)} \odot \cdots \odot \boldsymbol{u}_{\sigma(j_1)}\right) \odot g\left(\boldsymbol{u}_{\sigma(j_1+1)} \odot \cdots \odot \boldsymbol{u}_{\sigma(j_1+j_2)}\right), \tag{11}$$

where

$$S_{j_1,j_2} := \{\sigma \in \mathfrak{S}_{j_1+j_2} : \sigma(1) < \cdots < \sigma(j_1) \text{ and } \sigma(j_1+1) < \cdots < \sigma(j_1+j_2)\}. \tag{12}$$

Define

$$(\varphi_1 \times \varphi_2)(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{j_1+j_2}) = \left(\varphi_1(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{j_1}), \varphi_2(\boldsymbol{u}_{j_1+1}, \ldots, \boldsymbol{u}_{j_1+j_2})\right),$$

φ_i being the universal map of $\mathrm{Sym}^{j_i} V$; we intend the diagram involving the Cartesian product

$$\begin{tikzcd} V^{\times j_1+j_2} \arrow[d, "\varphi_1 \times \varphi_2"'] \arrow[dr, "\tilde{h}"] & \\ \mathrm{Sym}^{j_1} V \times \mathrm{Sym}^{j_2} V \arrow[r, "h"] & \mathrm{Sym}^{i_1+i_2} W \end{tikzcd} \tag{13}$$

to commute. Let $\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_{j_1+j_2}}$ be $j_1 + j_2$ vectors in V, each an element of base $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. We have

$$(\varphi_1 \times \varphi_2)\left(\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_{j_1+j_2}}\right) = \left(\boldsymbol{u}_{i_1} \odot \cdots \odot \boldsymbol{u}_{i_{j_1}}, \boldsymbol{u}_{i_{j_1+1}} \odot \cdots \odot \boldsymbol{u}_{i_{j_1+j_2}}\right)$$

and we may also split each set of vectors into copies of separate basis vectors:

$$\left\{\boldsymbol{u}_{i_1}, \ldots, \boldsymbol{u}_{i_{j_1}}\right\} = \{\boldsymbol{e}_1 \overset{p_1}{\ldots}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \overset{p_n}{\ldots}, \boldsymbol{e}_n\}, \quad \left\{\boldsymbol{u}_{i_{j_1+1}}, \ldots, \boldsymbol{u}_{i_{j_1+j_2}}\right\} = \{\boldsymbol{e}_1 \overset{q_1}{\ldots}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \overset{q_n}{\ldots}, \boldsymbol{e}_n\},$$

with $|\mathbf{p}| = j_1$ and $|\mathbf{q}| = j_2$, and define $\mathbf{k} = \mathbf{p} + \mathbf{q}$. The expression of (11) in these basis elements is now an immediate consequence of basic combinatorics:

$$\tilde{h}\left(\boldsymbol{e}_1 \overset{k_1}{\ldots}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n, \overset{k_n}{\ldots}, \boldsymbol{e}_n\right) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{|\mathbf{P}|=j_1} \binom{k_1}{P_1}\binom{k_2}{P_2} \cdots \binom{k_n}{P_n} f\left(\boldsymbol{e}^{\odot \mathbf{P}}\right) \odot g\left(\boldsymbol{e}^{\odot \mathbf{k}-\mathbf{P}}\right),$$

leaving no option for (13) to commute but

$$h\left(\boldsymbol{e}^{\odot \mathbf{p}}, \boldsymbol{e}^{\odot \mathbf{q}}\right) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{|\mathbf{P}|=j_1} \binom{p_1+q_1}{P_1}\binom{p_2+q_2}{P_2} \cdots \binom{p_n+q_n}{P_n} f\left(\boldsymbol{e}^{\odot \mathbf{P}}\right) \odot g\left(\boldsymbol{e}^{\odot \mathbf{p}+\mathbf{q}-\mathbf{P}}\right). \tag{14}$$

On the other hand, the universal property on the total symmetric product $\left(\mathrm{Sym}^{j_1+j_2}V, \widetilde{\varphi}\right)$ yields a unique $h_\odot$ such that $h_\odot \circ \widetilde{\varphi} \equiv \tilde{h}$,

$$\begin{tikzcd}
V^{\times j_1+j_2} \arrow[d, "\varphi_1\times\varphi_2"'] \arrow[dr, "\tilde{h}"] \arrow[dd, bend right=60, "\widetilde{\varphi}"'] & \\
\mathrm{Sym}^{j_1}V \times \mathrm{Sym}^{j_2}V \arrow[r, "h"] \arrow[d, "\varphi"] & \mathrm{Sym}^{i_1+i_2}W \\
\mathrm{Sym}^{j_1+j_2}V \arrow[ur, dashed, "h_\odot"] &
\end{tikzcd} \tag{15}$$

The fact $\varphi \circ (\varphi_1 \times \varphi_2) \equiv \widetilde{\varphi}$ is immediate. And fixing φ (and h) the uniqueness of $h_\odot$ follows from construction: any other $h_\bullet$ rendering (10) commutative would require the commutativity of the perimeter of (15), hence $h_\bullet \equiv h_\odot$.

Hence all we need to do is express $f \odot g := h_\odot$ in terms of its action on base elements (6) to obtain a simple, explicit form.

Notation 2.5. *When dealing with matrix sets, we will use super-indices and subindices in the following manner.*

1. *The space of (i,j)-**matrices** $\mathrm{Mat}^{i,j}_{m,n}(K)$ can either be defined by its underlying set, i.e. all $d_{m,i} \times d_{n,j}$ matrices having entries in K, or as the vector space of linear maps between symmetric powers $\mathrm{Hom}_K\left(\mathrm{Sym}^jV; \mathrm{Sym}^iW\right)$ whenever $V \cong K^m$ and $W \cong K^n$.*

2. *It is clear from the above that $\mathrm{Mat}^{0,0}_n(K)$ is the set of all scalars $\alpha \in K$ and $\mathrm{Mat}^{0,k}_n(K)$ (resp. $\mathrm{Mat}^{k,0}_n(K)$) is made up of all row (resp. column) vectors whose entries are indexed by $d_{n,k}$ lexicographically ordered k-tuples.*

2. *Reference to K may be dropped and notation may be abridged if dimensions are repeated or trivial, e.g. $\mathrm{Mat}^{i,j}_n := \mathrm{Mat}^{i,j}_{n,n}$, $\mathrm{Mat}^i_{m,n} := \mathrm{Mat}^{i,i}_{m,n}$, $\mathrm{Mat}_n := \mathrm{Mat}^1_n$, etcetera.*

Checking product $\odot$ defined below renders diagrams (10) and (15) commutative is now immediate.

Definition 2.6 (Symmetric product of finite matrices)**.** *Let $A \in \mathrm{Mat}_{m,n}^{i_1,j_1}(K)$, $B \in \mathrm{Mat}_{m,n}^{i_2,j_2}(K)$, i.e. linear maps $A : \mathrm{Sym}^{j_1} K^n \to \mathrm{Sym}^{i_1} K^m$ and $B : \mathrm{Sym}^{j_2} K^n \to \mathrm{Sym}^{i_2} K^m$.*

Given any multi-index $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$ and $|\mathbf{k}| = k_1 + \dots + k_n = j_1 + j_2$,

$$(A \odot B)\left(\boldsymbol{e}_1^{k_1} \cdots \boldsymbol{e}_n^{k_n}\right) = \frac{1}{\binom{j_1+j_2}{j_1}} \sum_{\mathbf{p}} \binom{\mathbf{k}}{\mathbf{p}} \left(A\boldsymbol{e}_1^{p_1} \cdots \boldsymbol{e}_n^{p_n}\right) \odot \left(B\boldsymbol{e}_1^{k_1-p_1} \cdots \boldsymbol{e}_n^{k_n-p_n}\right), \tag{16}$$

notation abused by removing $\odot$ to reduce space within basis elements (6), binomials as in (4) and summation taking place for specific multi-indices $\mathbf{p}$, namely those such that

$$|\mathbf{p}| = j_1 \qquad \text{and} \qquad 0 \leqslant p_i \leqslant k_i, \quad i = 1, \dots, n.$$

The following is a mere exercise in induction:

Lemma 2.7. *The product of $A_1, \dots, A_r$, recursively defined by*

$$A_1 \odot \cdots \odot A_r := (A_1 \odot \cdots \odot A_{r-1}) \odot A_r,$$

where $A_i \in \mathrm{Mat}_{m,n}^{k_i,j_i}$, $i = 1, \dots, r$, is expressed in terms of multinomials by

$$(A_1 \odot \cdots \odot A_r)\, \boldsymbol{e}^{\odot \mathbf{k}} = \frac{1}{\binom{j_1+\dots+j_r}{j_1,j_2,\dots,j_r}} \sum_{\mathbf{p}_1,\dots,\mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \dots, \mathbf{p}_r} \bigodot_{i=1}^{r} A_i \boldsymbol{e}^{\odot \mathbf{p}_i}, \quad \text{if} \quad |\mathbf{k}| = j_1 + \dots + j_r, \tag{17}$$

sums obviously taken for $\mathbf{p}_1 + \dots + \mathbf{p}_r = \mathbf{k}$ and $|\mathbf{p}_i| = j_i$, for every $i = 1, \dots, r$. □

Remarks 2.8.

1. For an equivalent "non-monic" formulation of (16) (i.e. one for which entry ${}_{1,1}$ need not have coefficient 1 in its formal expression) using multi-indices in both columns *and* rows, see e.g. [8, 9, 10, 11].

2. Notation in Proposition 2.3 extends to matrices: $\mathrm{Sym}^r A := A^{\odot r} := A \odot \overset{r}{\cdots} \odot A$.

3. In the case of a square $(1,1)$-matrix $A \in \mathrm{Mat}_n(K)$, powers ${}^{\odot r}$ according to (16) and (17) are obviously consistent with multiple product (8), hence equal to established definitions for group morphism $\mathrm{Sym}^r : \mathrm{GL}_n(V) \to \mathrm{GL}_n(\mathrm{Sym}^r(V))$ in multilinear algebra textbooks such as the expression in terms of the *permanent* of A (e.g. [13, Th. 9.2]), or $\frac{1}{r!} A \circledS \overset{r}{\cdots} \circledS A$ in multiple references such as [2, 5, 7].

Example 2.9. Given matrices $A \in \mathrm{Mat}_2^{1,1}(K)$ and $B \in \mathrm{Mat}_2^{3,2}(K)$, we may write them as

$$A = \left(\; A\boldsymbol{e}_1 \;\middle|\; A\boldsymbol{e}_2 \; \right) = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}, \quad B = \left(\; B\boldsymbol{e}_1^{\odot 2} \;\middle|\; B\boldsymbol{e}_1 \odot \boldsymbol{e}_2 \;\middle|\; B\boldsymbol{e}_2^{\odot 2} \; \right) = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \\ b_{4,1} & b_{4,2} & b_{4,3} \end{pmatrix}$$

and it is immediate to check that the $(4,3)$ (hence four-column, five-row) matrix product

$$A \odot B = \left(\; (A \odot B)\left(\boldsymbol{e}_1^{\odot 3}\right) \;\middle|\; (A \odot B)\left(\boldsymbol{e}_1^{\odot 2} \odot \boldsymbol{e}_2\right) \;\middle|\; (A \odot B)\left(\boldsymbol{e}_1 \odot \boldsymbol{e}_2^{\odot 2}\right) \;\middle|\; (A \odot B)\left(\boldsymbol{e}_2^{\odot 3}\right) \; \right)$$

is equal to

$$\begin{pmatrix} a_{1,1}b_{1,1} & \frac{1}{3}(a_{1,2}b_{1,1} + 2a_{1,1}b_{1,2}) & \frac{1}{3}(2a_{1,2}b_{1,2} + a_{1,1}b_{1,3}) & a_{1,2}b_{1,3} \\ a_{2,1}b_{1,1} + a_{1,1}b_{2,1} & \frac{a_{2,2}b_{1,1}+2a_{2,1}b_{1,2}+a_{1,2}b_{2,1}+2a_{1,1}b_{2,2}}{3} & \frac{2a_{2,2}b_{1,2}+a_{2,1}b_{1,3}+2a_{1,2}b_{2,2}+a_{1,1}b_{2,3}}{3} & a_{2,2}b_{1,3} + a_{1,2}b_{2,3} \\ a_{2,1}b_{2,1} + a_{1,1}b_{3,1} & \frac{a_{2,2}b_{2,1}+2a_{2,1}b_{2,2}+a_{1,2}b_{3,1}+2a_{1,1}b_{3,2}}{3} & \frac{2a_{2,2}b_{2,2}+a_{2,1}b_{2,3}+2a_{1,2}b_{3,2}+a_{1,1}b_{3,3}}{3} & a_{2,2}b_{2,3} + a_{1,2}b_{3,3} \\ a_{2,1}b_{3,1} + a_{1,1}b_{4,1} & \frac{a_{2,2}b_{3,1}+2a_{2,1}b_{3,2}+a_{1,2}b_{4,1}+2a_{1,1}b_{4,2}}{3} & \frac{2a_{2,2}b_{3,2}+a_{2,1}b_{3,3}+2a_{1,2}b_{4,2}+a_{1,1}b_{4,3}}{3} & a_{2,2}b_{3,3} + a_{1,2}b_{4,3} \\ a_{2,1}b_{4,1} & \frac{1}{3}(a_{2,2}b_{4,1} + 2a_{2,1}b_{4,2}) & \frac{1}{3}(2a_{2,2}b_{4,2} + a_{2,1}b_{4,3}) & a_{2,2}b_{4,3} \end{pmatrix}.$$

The following is straightforward to prove from either direct application of the universal property or the analogous techniques used in [8, 10], and will not be delved into here:

Proposition 2.10. *For any matrices A, B, C, whenever products make sense, the following properties hold:*

a) $A \odot B = B \odot A$.

b) $(A+B) \odot C = A \odot C + B \odot C$.

c) $(A \odot B) \odot C = A \odot (B \odot C)$.

d) $(\alpha A) \odot B = \alpha (A \odot B)$ *for every* $\alpha \in K$.

e) If A is square and invertible, then $(A^{-1})^{\odot k} = \left(A^{\odot k}\right)^{-1}$.

f) $A \odot B = 0$ *if and only if* $A = 0$ *or* $B = 0$.

g) If A is a square $(1,1)$-matrix, then $A\boldsymbol{v}_1 \odot A\boldsymbol{v}_2 \odot \cdots \odot A\boldsymbol{v}_m = A^{\odot m}\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_m$.

h) If $\boldsymbol{v}$ is a column vector, then $(A \odot B)\,\boldsymbol{v}^{\odot(p+q)} = (A\boldsymbol{v}^{\odot p}) \odot (B\boldsymbol{v}^{\odot q})$, *for every* $p, q \in \mathbb{Z}_{\geqslant 0}$.

i) If $\operatorname{rank}(A) = r$ *then* $\operatorname{rank}(A^{\odot m}) = d_{r,m}$ *and* $\det A^{\odot m} = (\det A)^{\binom{m+n-1}{n}}$. □

The next two results are immediate as well:

Lemma 2.11. *For any two matrices $A \in \mathrm{Mat}_n^{i,j}$ and $B \in \mathrm{Mat}_n^{p,q}$ and vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j+q} \in V$, the following holds, $S_{j,q}$ defined as in (12):*

$$(A \odot B)(\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_{j+q}) = \frac{1}{\binom{j+q}{q}} \sum_{\sigma \in S_{j,q}} A\left(\boldsymbol{v}_{\sigma(1)} \odot \cdots \odot \boldsymbol{v}_{\sigma(j)}\right) \odot B\left(\boldsymbol{v}_{\sigma(j+1)} \odot \cdots \odot \boldsymbol{v}_{\sigma(j+q)}\right). \tag{18}$$

Proof. Nothing but the universal property on (11) and diagram (15) with different notation. □

Lemma 2.12. *Let $k \geqslant 1$ and $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be a basis of K^n.*

a) (see also [8, 10]) For any $A \in \mathrm{Mat}_n^{p,q}$ and $B \in \mathrm{Mat}_n^{q,r}$, and every vector $\boldsymbol{v} \in \mathrm{Sym}^k K^n$,

$$(\boldsymbol{v} \odot A)\, B = \boldsymbol{v} \odot (AB)\,. \tag{19}$$

b) If $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is the canonical basis formed by columns in Id_n, then

$$\sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right)\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \mathrm{Id}_n^{\odot k+1}.$$

Proof. a) It suffices to prove it for basis elements of Sym^q: for any $\mathbf{k}$ such that $|\mathbf{k}| = p$,

$$(\boldsymbol{v} \odot A)\, B\boldsymbol{e}^{\odot \mathbf{k}} = \boldsymbol{v} \odot (AB)\, \boldsymbol{e}^{\odot \mathbf{k}}. \tag{20}$$

But this is immediate from equation (18) or the definition (16) of $\odot$ itself.

b) Using the previous item and the associative property in Proposition 2.10,

$$\left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right)\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \boldsymbol{e}_m \odot \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \left(\boldsymbol{e}_m \odot \boldsymbol{e}_m^T\right) \odot \mathrm{Id}_n^{\odot k-1};$$

it is immediate to check that $\boldsymbol{e}_m \odot \boldsymbol{e}_m^T$ is a square $(2,2)$-matrix whose only non-zero element is 1 in position ${}_{m,m}$, hence

$$\sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right)\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \left[\sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \boldsymbol{e}_m^T\right)\right] \odot \mathrm{Id}_n^{\odot k-1} = \mathrm{Id}_n^{\odot 2} \odot \mathrm{Id}_n^{\odot k-1} = \mathrm{Id}_n^{\odot k+1}.$$

□

2.2 More properties of $\odot$

We need to generalise some of the above properties in Proposition 2.10 for later purposes. Matrices will not be necessarily square unless specifically defined as such throughout this Section.

Lemma 2.13. *Given square matrices $A, B \in \mathrm{Mat}_n^{k,k}$ and matrices $X_i \in \mathrm{Mat}_n^{k,j_i}$, $i = 1, 2$,*

$$(A \odot B)(X_1 \odot X_2) = \frac{1}{2}(AX_1 \odot BX_2 + BX_1 \odot AX_2), \tag{21}$$

and in general for any square $A_1, \ldots, A_m \in \mathrm{Mat}_n^{k,k}(K)$ and any $X_i \in \mathrm{Mat}_n^{k,j_i}(K)$, $i = 1, \ldots, m$,

$$\left(\bigodot_{i=1}^m A_i\right)\left(\bigodot_{i=1}^m X_i\right) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_k}\bigodot_{i=1}^m A_{\sigma(i)}X_i. \tag{22}$$

Proof. All it takes for $m = 2$ is applying the universal property on either (7) (replacing V with $\mathrm{Sym}^k K^n$) or (14) (replacing j_1 and j_2 by k) on the product of $A \odot B$ by each of the columns of $X_1 \odot X_2$: indeed, for every $\mathbf{i}$ such that $|\mathbf{i}| = j_1 + j_2$, $X_1\boldsymbol{e}^{\odot\mathbf{p}}$ and $X_2\boldsymbol{e}^{\odot\mathbf{i}-\mathbf{p}}$ are both vectors of $\mathrm{Sym}^k K^n$ whenever $|\mathbf{i}| = j_1$, hence

$$(A \odot B)\left[\left(X_1\boldsymbol{e}^{\odot\mathbf{p}}\right) \odot \left(X_2\boldsymbol{e}^{\odot\mathbf{i}-\mathbf{p}}\right)\right] = \frac{1}{2}\left[A\left(X_1\boldsymbol{e}^{\odot\mathbf{p}}\right) \odot B\left(X_2\boldsymbol{e}^{\odot\mathbf{i}-\mathbf{p}}\right) + B\left(X_1\boldsymbol{e}^{\odot\mathbf{p}}\right) \odot A\left(X_2\boldsymbol{e}^{\odot\mathbf{i}-\mathbf{p}}\right)\right],$$

and attachment of $\frac{1}{\binom{j_1+j_2}{j_1}}\sum_{|\mathbf{p}|=j_1}\binom{\mathbf{i}}{\mathbf{p}}$ to both sides of the equation, along with (16), yields (21).

Equally true for arbitrary m, using the universal property on the multiple product (8), now expressed on $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ as $\left(\bigodot_i A_i\right)\left(\bigodot_i \boldsymbol{v}_i\right) = \frac{1}{m!}\sum_{\sigma\in\mathfrak{S}_m}\bigodot_i A_i\boldsymbol{v}_{\sigma(i)}$, replacing each $\boldsymbol{v}_i$ by the corresponding product $\bigodot_{i=1}^m A_i\boldsymbol{e}^{\odot\mathbf{p}_i}$ and attaching $\binom{j_1+\cdots+j_m}{j_1,j_2,\ldots,j_m}^{-1}\sum_{\mathbf{p}_1,\ldots,\mathbf{p}_m}\binom{\mathbf{k}}{\mathbf{p}_1,\ldots,\mathbf{p}_m}$ as in (17). □

Lemma 2.14. *Given $A \in \mathrm{Mat}_n^{1,j}$ and $X_1, \ldots, X_m$ such that $X_i \in \mathrm{Mat}_n^{1,q_i}$, assuming $1 \leqslant j \leqslant m$,*

$$\binom{m}{j}\left(A \odot \mathrm{Id}_n^{\odot m-j}\right)\left(\bigodot_{i=1}^m X_i\right) = \sum_{1\leqslant i_1<\cdots<i_j\leqslant m}\left[A\left(X_{i_1} \odot \cdots \odot X_{i_j}\right)\right] \odot \bigodot_{s\neq i_1,\ldots,i_j} X_s. \tag{23}$$

Proof. Defining $B := \mathrm{Id}_n^{\odot m-j} \in \mathrm{Mat}_n^{m-j,m-j}$ and $\boldsymbol{v}_i := X_i\boldsymbol{e}^{\odot\mathbf{p}_i}$ where $|\mathbf{p}_i| = q_i$, $i = 1, \ldots, m$, equation (18) becomes

$$\left(A \odot \mathrm{Id}_n^{\odot m-j}\right)\bigodot_{r=1}^m X_r\boldsymbol{e}^{\odot\mathbf{p}_r} = \frac{1}{\binom{m}{j}}\sum_{1\leqslant i_1<\cdots<i_j\leqslant m} A\left(X_{i_1}\boldsymbol{e}^{\odot\mathbf{p}_{i_1}} \odot \cdots \odot X_{i_j}\boldsymbol{e}^{\odot\mathbf{p}_{i_j}}\right) \odot \bigodot_{s\neq i_1,\ldots,ij}^m X_s\boldsymbol{e}^{\odot\mathbf{p}_s}. \tag{24}$$

Attach $\binom{q_1+\cdots+q_m}{q_1,q_2,\ldots,q_m}^{-1}\sum_{\mathbf{p}_1,\ldots,\mathbf{p}_m}\binom{\mathbf{k}}{\mathbf{p}_1,\ldots,\mathbf{p}_m}$ to both sides of the equation and let $\mathbf{k}$ be any ordered multi-index having modulus $q_1 + \cdots + q_m$. The left-hand side becomes $\left(A \odot \mathrm{Id}_n^{\odot m-j}\right)\left(\bigodot_{i=1}^m X_i\right)$. Multiplying the right-hand side of (23) by $\boldsymbol{e}^{\mathbf{k}}$ and expressing the result in terms of each of its symmetric product factors, the product of the resulting five binomial and multinomial coefficients is equal to $\binom{q_1+\cdots+q_m}{q_1,q_2,\ldots,q_m}^{-1}$ and the span of the multi-indices is precisely that of those in the right-hand side of (24) once embedded into its sum. □

An immediate consequence of either Lemma 2.13 or Lemma 2.14 is

Corollary 2.15. *Given a square matrix $A \in \mathrm{Mat}_n^{1,1}$ and $X_1, \ldots, X_m$ such that $X_i \in \mathrm{Mat}_n^{1,j_i}$,*

$$\left(A \odot \mathrm{Id}_n^{\odot m-1}\right)\left(\bigodot_{i=1}^m X_i\right) = \frac{1}{m}\sum_{i=1}^m (AX_i) \odot \left(X_1 \odot \cdots \odot \widehat{X_i} \odot \cdots \odot X_m\right). \qquad \square \tag{25}$$

Lemma 2.16. *Given square matrix $X \in \mathrm{Mat}_n^{1,1}$, any vector $\boldsymbol{v} \in K^n$ and $r \geqslant 1$,*

$$\left(X\boldsymbol{v} \odot \mathrm{Id}^{\odot r}\right) X^{\odot r} = X^{\odot r+1} \left(\boldsymbol{v} \odot \mathrm{Id}^{\odot r}\right). \tag{26}$$

Proof. The first step is proving $\left(X\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) X^{\odot r} = X\boldsymbol{v} \odot X^{\odot r}$. This is immediate from (19) but let us elaborate on the proof for the sake of clarity and illustration. The fact $\mathrm{Id}_n^{\odot r} = \mathrm{Id}_{d_{n,r}}$ simplifies some steps. Equation (18) for $i = 1$, $j = 0$, $p = q = r$ yields, for every set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in K^n$,

$$\left(X\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) \left(\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_r\right) = X\boldsymbol{v} \odot \left(\boldsymbol{v}_1 \odot \cdots \odot \boldsymbol{v}_r\right). \tag{27}$$

Set $\boldsymbol{v}_i := X\boldsymbol{e}^{\odot \mathbf{p}_i} \in K^n$ for any given $\mathbf{p}_i$, slight notation abuse notwithstanding since $|\mathbf{p}_i| = 1$. Then (27) becomes

$$\left(X\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) \left(X\boldsymbol{e}^{\odot \mathbf{p}_1} \odot \cdots \odot X\boldsymbol{e}^{\odot \mathbf{p}_r}\right) = X\boldsymbol{v} \odot X\boldsymbol{e}^{\odot \mathbf{p}_1} \odot \cdots \odot X\boldsymbol{e}^{\odot \mathbf{p}_r}. \tag{28}$$

Hence

$$\begin{aligned}
\left(X\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) X^{\odot r} \boldsymbol{e}^{\odot \mathbf{k}} &= \left(X\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) \frac{1}{r!} \sum_{\mathbf{p}_1, \ldots, \mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \ldots, \mathbf{p}_r} \left(X\boldsymbol{e}^{\odot \mathbf{p}_1} \odot \cdots \odot X\boldsymbol{e}^{\odot \mathbf{p}_r}\right) \\
&= \frac{1}{r!} \sum_{\mathbf{p}_1, \ldots, \mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \ldots, \mathbf{p}_r} X\boldsymbol{v} \odot X\boldsymbol{e}^{\odot \mathbf{p}_1} \odot \cdots \odot X\boldsymbol{e}^{\odot \mathbf{p}_r} \\
&= X\boldsymbol{v} \odot \frac{1}{r!} \sum_{\mathbf{p}_1, \ldots, \mathbf{p}_r} \binom{\mathbf{k}}{\mathbf{p}_1, \ldots, \mathbf{p}_r} X\boldsymbol{e}^{\odot \mathbf{p}_1} \odot \cdots \odot X\boldsymbol{e}^{\odot \mathbf{p}_r} \\
&= X\boldsymbol{v} \odot X^{\odot r} \boldsymbol{e}^{\odot \mathbf{k}}
\end{aligned} \tag{29}$$

sum multi-indices $\mathbf{p}_i$ adding up to $\mathbf{k}$ and having successive moduli j_i as always.

Equation (20) and application of Proposition 2.10 imply

$$X^{\odot r+1} \left(\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) \boldsymbol{e}^{\odot \mathbf{k}} = X^{\odot r+1} \left(\boldsymbol{v} \odot \boldsymbol{e}^{\odot \mathbf{k}}\right) = X\boldsymbol{v} \odot X^{\odot k_1} \boldsymbol{e}_1^{\odot k_1} \odot \cdots \odot X^{\odot k_1} \boldsymbol{e}_n^{\odot k_n},$$

precisely (29). ☐

If (K, ∂) is a *differential field* [28], i.e. $\partial : K \to K$ is a *derivation*, a linear map satisfying the Leibniz rule $\partial(ab) = b\partial(a) + a\partial(b)$, this extends entrywise to matrices, $\partial(a_{i,j}) := (\partial a_{i,j})$ and the Leibniz rule applies to $\odot$. This is immediate and well-known for square matrices, but we are in a more general case:

Lemma 2.17. *For any given $X \in \mathrm{Mat}_n^{k_1,j_1}(K)$ and $Y \in \mathrm{Mat}_n^{k_2,j_2}(K)$,*

$$\partial(X \odot Y) = \partial(X) \odot Y + X \odot \partial(Y). \tag{30}$$

Proof. It suffices, from expression (16), to check it true for symmetric products of *vectors* $\boldsymbol{u}, \boldsymbol{v} \in \mathrm{Sym}^{k_1+k_2}(K^n)$ but this is as trivial as for homogeneous polynomials in n arbitrary unknowns $K[E_1, \ldots, E_n]$ in virtue of Proposition 2.3 or isomorphism $\mathrm{Sym}^{k_1+k_2}(V^\star) \cong S^{k_1+k_2}(V, K)$ and can be gleaned from any of references mentioned in Section 2. ☐

Although the next result will be rendered academic by simplified expressions in Section 4.1, it is worth writing for the sake of clarifying certain routinely-appearing matrix blocks a bit further.

Lemma 2.18.

a) If Y is a square $n \times n$ matrix having entries in K and $\dot{Y} = AY$, then

$$\frac{d}{dt}\mathrm{Sym}^k Y = k\left(A \odot \mathrm{Sym}^{k-1}(\mathrm{Id}_n)\right)\mathrm{Sym}^k Y. \tag{31}$$

b) If $X \in \mathrm{Mat}_n^{1,j_1}(K)$ and $Y \in \mathrm{Mat}_n^{1,j_2}(K)$ satisfy systems

$$\dot{X} = AX + B_1, \quad \dot{Y} = AY + B_2, \qquad\qquad A \in \mathrm{Mat}_n^{1,1}(K), B_i \in \mathrm{Mat}_n^{1,j_i}(K),$$

then the symmetric product of these matrices satisfies the linear system

$$\frac{d}{dt}(X \odot Y) = 2\left(A \odot \mathrm{Id}_{d_{n,k}}\right)(X \odot Y) + (B_1 \odot Y + B_2 \odot X). \tag{32}$$

c) More generally, if $X_i, B_i \in \mathrm{Mat}_n^{1,j_i}(K)$, $i = 1, \ldots, m$, $A \in \mathrm{Mat}_n^{1,1}$ and

$$\dot{X}_i = AX_i + B_i, \qquad i = 1, \ldots, m,$$

then

$$\frac{d}{dt}\bigodot_{i=1}^m X_i = m\left(A \odot \mathrm{Id}_{d_{n,k}}^{\odot m-1}\right)\bigodot_{i=1}^m X_i + \sum_{i=1}^m B_i \odot \bigodot_{j \neq i} X_j. \tag{33}$$

Proof. a) Immediate upon application of commutativity and (22) or (25) to

$$\frac{d}{dt}\left(Y \odot \overset{k)}{\cdots} \odot Y\right) = \sum_{i=1}^k Y \odot \cdots \odot \overbrace{\dot{Y}}^{k} \odot \cdots \odot Y = \sum_{i=1}^k Y \odot \cdots \odot \overbrace{AY}^{k} \odot \cdots \odot Y.$$

b) Follows from Lemma 2.17 and the commutative and distributive properties of $\odot$.

c) We use induction, $m = 2$ being the previous item. Leibniz rule (30) and associativity imply

$$\dot{\overline{\bigodot_{i=1}^m X_i}} = \left(\frac{d}{dt}\bigodot_{i=1}^{m-1} X_i\right) \odot X_m + \left(\bigodot_{i=1}^{m-1} X_i\right) \odot \dot{X}_m,$$

wherein induction hypothesis implies

$$\left[(m-1)\left(A \odot \mathrm{Id}_{d_{n,k}}^{\odot m-1}\right)\bigodot_{i=1}^{m-1} X_i + \sum_{i=1}^{m-1} B_i \odot \bigodot_{j \neq i} X_j\right] \odot X_m + (AX_m + B_m) \odot \bigodot_{i=1}^{m-1} X_i,$$

which is equal in virtue of the distributive property and (25) to

$$\begin{aligned}&\left[\sum_{i=1}^{m-1} AX_i \odot \left(X_1 \odot \cdots \odot \widehat{X_i} \odot \cdots \odot X_{m-1}\right)\right] \odot X_m + \sum_{i=1}^{m-1} B_i \odot \bigodot_{j \neq i} X_j \odot X_m \\ &+ AX_m \odot \bigodot_{i=1}^{m-1} X_i + B_m \odot \bigodot_{i=1}^{m-1} X_i \\ &= \left[\sum_{i=1}^m AX_i \odot \left(X_1 \odot \cdots \odot \widehat{X_i} \odot \cdots \odot X_m\right)\right] + \sum_{i=1}^m B_i \odot \bigodot_{j \neq i} X_j,\end{aligned}$$

further application of (25) ending the proof. □

Remark 2.19. Albeit not explicitly as in (31), the matrix proven equal to $k\left(A \odot \mathrm{Id}_n^{\odot k-1}\right)$ has appeared in numerous prior references (e.g. [2, 3, 4, 5, 7]) whenever an equation for Sym^k such as (31) arises, has been sometimes labelled $\mathrm{sym}^k A$ and has been consistently called symmetric power of A *in the sense of Lie algebras*, its *Lie group* counterpart summarily standing therein for ${}^{\odot k}$ as defined in this paper.

3 Symmetric products and exponentials of infinite matrices

The next step towards a compact form to linearised higher variationals is assembling the matrix blocks gleaned from Remark 2.19 together into a compact matrix whenever dealing with different blocks $Y_1, Y_2, \ldots$ satisfying different differential equations.

3.1 Products and exponentials

Of the myriad ways to note a set of infinite matrices, we may need one taking finite submatrix orders into account. Alternatively, of all the ways in which to write a K-algebra S, a need may arise to express it whenever possible $S = \mathrm{Sym}(V) := \bigoplus_{k \geqslant 0} \mathrm{Sym}^k(V)$ for a given vector space.

Notation 3.1. *From now on* $\mathrm{Mat}^{n,m}(K)$ *denotes the set of all block matrices*

$$A = (A_{i,j})_{i,j \geqslant 0}, \qquad A_{i,j} : \mathrm{Sym}^i K^m \to \mathrm{Sym}^j K^n,$$

hence $A_{i,j} \in \mathrm{M}_{d_{n,i} \times d_{m,j}}(K) = \mathrm{Mat}^{i,j}_{n,m}(K)$*:*

$$A = \left(\begin{array}{c|c|c|c|c}
\ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
\cdots & A_{3,3} & A_{3,2} & A_{3,1} & \leftarrow A_{3,0} \\ \hline
\cdots & A_{2,3} & A_{2,2} & A_{2,1} & \leftarrow A_{2,0} \\ \hline
\cdots & A_{1,3} & A_{1,2} & A_{1,1} & \leftarrow A_{1,0} \\ \hline
\cdots & A_{0,3} & A_{0,2} & A_{0,1} & \leftarrow A_{0,0}
\end{array}\right)$$

We write $\mathrm{Mat}(K) := \mathrm{Mat}^{n,n}(K)$ *if the context allows for it, i.e. the value of* n *is unambiguous.*

Conversely, the set of matrices $\mathrm{Mat}^{i,j}_{n,m}(K)$ *is identified as a subset of* $\mathrm{Mat}^{n,m}(K)$ *by identifying every such block* $A_{i,j}$ *with an element of* $\mathrm{Mat}^{n,m}(K)$ *all of whose blocks are zero save for, perhaps,* $A_{i,j}$*.*

We define the following product on $\mathrm{Mat}^{n,m}(K)$. For a formulation yielding the same results see [11, p. 2].

Definition 3.2. *We define, for every given $A, B \in \mathrm{Mat}^{n,m}(K)$, matrix $A \odot B = C \in \mathrm{Mat}^{n,m}(K)$ where for every given i, j,*

$$C_{i,j} = \sum_{\substack{0 \leqslant i_1 \leqslant i \\ 0 \leqslant j_1 \leqslant j}} \binom{j}{j_1} A_{i_1,j_1} \odot B_{i-i_1,j-j_1}. \tag{34}$$

Same as always, $^{\odot k}$ will stand for powers built with this product.

Example 3.3. Matrix $A \odot B$ takes the following form in its simplest echelons:

$$\left(\begin{array}{cc|c} \ddots & \vdots & \vdots \\ \ddots & \vdots & A_{0,0}B_{2,0} + A_{1,0} \odot B_{1,0} + A_{2,0}B_{0,0} \\ \cdots & A_{0,0}B_{1,1} + A_{0,1} \odot B_{1,0} + A_{1,0} \odot B_{0,1} + A_{1,1}B_{0,0} & A_{0,0}B_{1,0} + A_{1,0}B_{0,0} \\ \hline \cdots & A_{0,0}B_{0,1} + A_{0,1}B_{0,0} & A_{0,0}B_{0,0} \end{array}\right),$$

and coefficients other than 1 will start appearing in further block rows and columns. We split row $_{\star,0}$ and column $_{0,\star}$ from the rest of the matrix for further clarity.

The following is immediate and part of it has already been mentioned before, e.g. [10]:

Lemma 3.4. *Product $\odot$ on $\mathrm{Mat}^{n,m}$ is associative and commutative, and $(\mathrm{Mat}(K), +, \odot)$ is an integral domain as well as a K-algebra if endowed with the usual product by scalars in K. Its identity element is 1 made up of zero blocks except for $_{0,0}$ which is equal to 1_K.* $\square$

Definition 3.5. *For every matrix $A \in \mathrm{Mat}(K)$ we define the formal power series*

$$\exp_\odot A := 1 + A^{\odot 1} + \frac{1}{2} A^{\odot 2} + \cdots = \sum_{i=0}^{\infty} \frac{1}{i!} A^{\odot i}.$$

Whenever A has all but a finite distinguished submatrix $A_{j,k}$ equal to zero (e.g. Examples 3.7 below or Lemma 3.11), the abuse of notation $\exp_\odot A_{j,k} = \exp_\odot A$ will be customary.

See also [10]. The fact $\odot$ is commutative saves us the trouble of having to check matrix commutation in the properties below, whose proof works exactly like that of scalar exponentials:

Lemma 3.6.

a) For every two $A, B \in \mathrm{Mat}^{n,m}$, $\exp_\odot (A + B) = \exp_\odot A \odot \exp_\odot B$.

b) For every $Y \in \mathrm{Mat}^{n,m}$ and any derivation $\delta : K \to K$ defined on field K, then

$$\delta \exp_\odot Y = (\delta Y) \odot \exp_\odot Y.$$

c) ([8, Corollary 3]) For every two square matrices $A, B \in \mathrm{Mat}_n^{1,1}$, $\exp_\odot AB = \exp_\odot A \exp_\odot B$.

d) In particular, for every invertible square matrix $A \in \mathrm{Mat}_n^{1,1}$, $\exp_\odot A^{-1} = (\exp_\odot A)^{-1}$. $\square$

Examples 3.7.

1. Let $A \in \mathrm{Mat}(K)$ such that all blocks are zero except for $_{1,1}$:

$$A = \left(\begin{array}{cc|c} \ddots & \vdots & \vdots \\ \cdots & A_{1,1} & 0 \\ \hline \cdots & 0 & 0 \end{array}\right).$$

For $\frac{1}{2}A^{\odot 2}$, expressions (34) containing $A_{1,1}^{\odot 2}$ are those for which $i = j = 2$, hence

$$1 + A + \frac{1}{2}A^{\odot 2} = \left(\begin{array}{cccc|c} \ddots & \vdots & \vdots & & \vdots \\ \cdots & A_{1,1}^{\odot 2} & & & \\ \cdots & & A_{1,1} & & \\ \hline \cdots & & & & 1 \end{array}\right)$$

and the pattern is clear in general: $\exp_\odot A_{1,1} = \exp_\odot A = \operatorname{diag}\left(\cdots, A_{1,1}^{\odot k}, \ldots, A_{1,1}, 1\right)$.

2. For row or column vectors this expression is even simpler. If the only non-zero block in A is a row vector, $A_{k,0} = \boldsymbol{x} = \left(x_1, \ldots, x_{d_{n,k}}\right) \in \operatorname{Mat}_{n,k,0}(K)$,

$$A = \left(\begin{array}{cccc|c} \ddots & \vdots & \vdots & & \vdots \\ \cdots & 0 & 0 & \cdots & 0 \\ \hline \cdots & 0 & \boldsymbol{x} & \cdots & 0 \end{array}\right),$$

the only expression (34) not automatically zero in $(A \odot A)$ is $(A \odot A)_{0,2k} = \binom{2k}{k} A_{0,k} \odot A_{0,k}$. Recursively, the only expression not automatically zero in $A^{\odot j}$ is

$$\left(A \odot A^{j-1}\right)_{0,jk} = \binom{jk}{k} \cdot \binom{(j-1)\,k}{k} \cdots \binom{2k}{k} A_{0,k}^{\odot j}. \tag{35}$$

For instance, if $k = 1$,

$$\exp_\odot A = \sum_{j \geqslant 0} \boldsymbol{x}^{\odot j} = \left(\begin{array}{ccccc|c} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 0 \\ \hline \cdots & \boldsymbol{x}^{\odot 4} & \boldsymbol{x}^{\odot 3} & \boldsymbol{x}^{\odot 2} & \boldsymbol{x} & 1 \end{array}\right).$$

3. This does not apply *mutatis mutandis* to matrices whose only non-trivial blocks are in the ${}_{0,k}$ column. The only non-trivial block in $A^{\odot j}$ is ${}_{jk,0}$ whose expression is summarised in switching row and column indices and expunging binomials from (35). For $k = 1$, we have

$$\exp_\odot \boldsymbol{x} = \exp_\odot \left(\begin{array}{cc|c} \ddots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & \boldsymbol{x} \\ \hline \cdots & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} \ddots & \vdots & \vdots \\ \cdots & 0 & \frac{1}{j!}\boldsymbol{x}^{\odot j} \\ & \vdots & \vdots \\ \cdots & 0 & \frac{1}{2}\boldsymbol{x}^{\odot 2} \\ \cdots & 0 & \boldsymbol{x} \\ \hline \cdots & 0 & 1 \end{array}\right).$$

A fourth example, namely matrices $A \in \operatorname{Mat}(K)$ equal to zero save for block row ${}_{1,k}$, $k \geqslant 1$ (see (37)), deserves special attention in the forthcoming Sections. Let us first fix conventions:

Notation 3.8. *For every set of indices satisfying* $1 \leqslant i_1 \leqslant \cdots \leqslant i_r$ *and* $i_1 + \cdots + i_r = k$, $c^k_{i_1,\ldots,i_r}$ *is defined as the amount of totally ordered partitions of a set of* k *elements among subsets of sizes* $i_1, \ldots, i_r$*. We will write* $c^k_{\mathbf{i}}$ *following* $\mathbf{i} = (i_1, \ldots, i_r)$ *and omit super-index* ^{k} *if sum* $|\mathbf{i}|$ *is known beforehand.*

Remarks 3.9.

1. $c^k_{i_1,\ldots,i_j} = \# I^{i_1,\ldots,i_j}_{1,\ldots,k}$ following (61) below. Needless to say, $\sum_{i_1+\cdots+i_j=k} c^k_{i_1,\ldots,i_j} = \left\{ {k \atop j} \right\}$, the *Stirling number of the second type* ([1, §24.1.4]), and $\sum_{j=1}^k \sum_{i_1+\cdots+i_j=k} c^k_{i_1,\ldots,i_j} = B_k$, the k^{th} *Bell number* [27, Vol 2, Ch. 3].

2. Since each subset of size i_s is supposed to be ordered, we must divide the total amount by the orders of the corresponding symmetric groups, hence the explicit formula:

$$c^k_{i_1,\dots,i_j} = \frac{\binom{k}{i_1\, i_2 \cdots i_j}}{n_1! \cdots n_m!}, \quad \text{where} \begin{cases} (i_1,\dots,i_j) = (k_1 \overset{n_1}{\cdots} k_1, k_2 \overset{n_2}{\cdots} k_2, \cdots, k_m \overset{n_m}{\cdots} k_m), \\ 1 \leqslant k_1 < k_2 < \cdots < k_m. \end{cases} \tag{36}$$

Lemma 3.10. *Let $Y \in \mathrm{Mat}\,(K)$ equal to zero outside of block row $_{1,k}$, $k \geqslant 1$:*

$$Y := \left(\begin{array}{cccc|c} & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & Y_3 & Y_2 & Y_1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad Y_i \in \mathrm{Mat}^{1,i}_n. \tag{37}$$

Let $Z_{r,s}$, $s, r \geqslant 1$, be the corresponding block in $\exp_\odot Y$. Then,

a) Row block r in $\exp_\odot Y$ is recursively obtained in terms of row blocks 1 and $r-1$:

$$Z_{r,s} = \frac{1}{r} \sum_{j=1}^{s-r+1} \binom{s}{j} Y_j \odot Z_{r-1,s-j}. \tag{38}$$

In particular, $Z_{r,r} = Y_1^{\odot r}$ and $Z_{r,s} = 0_{d_{n,r},d_{n,s}}$ whenever $r > s$.

b) For every $m, r \geqslant 1$ and any $\boldsymbol{v} \in K^n$,

$$\left(Y_1 \boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right) Z_{r,r} = Z_{r+1,r+1} \left(\boldsymbol{v} \odot \mathrm{Id}_n^{\odot r}\right). \tag{39}$$

c) Using Notation 3.8 and (36), for every $s \geqslant r$

$$Z_{r,s} = \sum_{i_1+\cdots+i_r=s} c^s_{i_1,\dots,i_r} Y_{i_1} \odot Y_{i_2} \odot \cdots \odot Y_{i_r}. \tag{40}$$

d) Let $A \in \mathrm{Mat}\,(K)$ defined with similar disposition as Y, its horizontal strip not necessarily at level $_{1,*}$:

$$A := \left(\begin{array}{cccc|c} & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & A_3 & A_2 & A_1 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad A_t \in \mathrm{Mat}^{p,t}_n.$$

For every $t, i \geqslant 1$ and $s \geqslant t+i$, the following factorization holds:

$$\sum_{j=t}^{s-i} \binom{s}{j} (A_t Z_{t,j}) \odot Z_{i,s-j} = \binom{t+i}{i} \left(A_t \odot \mathrm{Id}_n^{\odot i}\right) Z_{t+i,s}. \tag{41}$$

e) If $Q \in \mathrm{Mat}^n$ has only its square $_{1,1}$ block different from zero, then $\exp_\odot QY = (\exp_\odot Q)(\exp_\odot Y)$.

Proof. a) Using (34) on $A = Y$ and $B = Y^{\odot s-1}$,

$$Z_{r,s} = \frac{1}{r!} \sum_{1 \leqslant s_1 \leqslant s} \binom{s}{s_1} A_{1,s_1} \odot B_{r-1,s-s_1} = \frac{1}{r!} \sum_{1 \leqslant s_1 \leqslant s} \binom{s}{s_1} Y_{s_1} \odot ((r-1)! Z_{r-1,s-s_1});$$

hence, using the fact $Z_{i,j} = 0$ for $i > j$, (38) ensues.

b) Direct from (26) in Lemma 2.16.

c) By induction. For $s=1$, r can only be equal to 1 in order to have a non-zero block and $Z_{1,1}=Y_1=c_1^1Y_1$. Assume (40) holds for all r smaller than or equal to $s-1$. We have

$$Z_{r,s}=\frac{1}{r}\sum_{j=1}^{s-r+1}\binom{s}{j}Y_j\odot Z_{r-1,s-j}=\frac{1}{r}\sum_{j=1}^{s-r+1}\binom{s}{j}Y_j\odot\sum_{i_1+\cdots+i_{r-1}=s-j}c^{s-j}_{i_1,\ldots,i_{r-1}}Y_{i_1}\odot Y_{i_2}\odot\cdots\odot Y_{i_{r-1}}.$$

Summand redistribution renders the above equal to

$$\frac{1}{r}\sum_{j_1+\cdots+j_r=s}C_{j_1,\ldots,j_r}Y_{j_1}\odot Y_{j_2}\odot\cdots\odot Y_{j_r},\tag{42}$$

where, setting $(j_1,\ldots,j_r)=(k_1\overset{n_1}{\cdots}k_1,k_2\overset{n_2}{\cdots}k_2,\cdots,k_m\overset{n_m}{\cdots}k_m)$, and defining $\mathbf{k}_i:=(k_i\overset{n_i}{\cdots}k_i)$ and $\mathbf{K}_i:=\left(\mathbf{k}_1,\ldots,\mathbf{k}_{i-1},k_i\overset{n_i-1}{\cdots}k_i,\mathbf{k}_{i+1},\ldots,\mathbf{k}_m\right)$,

$$C_{j_1,\ldots,j_r}=C_{\mathbf{k}_1,\ldots,\mathbf{k}_m}:=\binom{s}{k_1}c^{s-k_1}_{\mathbf{K}_1}+\binom{s}{k_2}c^{s-k_2}_{\mathbf{K}_2}+\cdots+\binom{s}{k_m}c^{s-k_m}_{\mathbf{K}_m}.$$

Each of the summands in $C_{j_1,\ldots,j_r}$ is equal to

$$\frac{s!}{k_i!\,(s-k_i)!}\,\frac{\frac{(s-k_i)!}{k_1!^{n_1}k_2!^{n_2}\cdots k_{i-1}!^{n_{i-1}}k_i!^{n_i}k_{i+1}!^{n_{i+1}}\cdots k_m!^{n_m}}}{n_1!n_2!\cdots(n_i-1)!\cdots n_m!}=n_m c^s_{j_1,\ldots,j_r},$$

hence the coefficient of $Y_{j_1}\odot\cdots\odot Y_{j_r}$ in (42) is equal to $\frac{1}{r}\left(n_1+\cdots+n_m\right)c^s_{j_1,\ldots,j_r}=c^s_{j_1,\ldots,j_r}$.

d) Let us express the left-hand side in (41) in terms of (40):

$$\sum_{j=t}^{s-i}\binom{s}{j}\left[A_t\sum_{m_1+\cdots+m_t=j}c^j_{m_1,\ldots,m_t}Y_{m_1}\odot\cdots\odot Y_{m_t}\right]\odot\left[\sum_{k_1+\cdots+k_i=s-j}c^{s-j}_{k_1,\ldots,k_i}Y_{k_1}\odot\cdots\odot Y_{k_i}\right],$$

distributivity yielding it equal to

$$\sum_{j=t}^{s-i}\binom{s}{j}\sum_{m_1,\ldots,m_t}\sum_{k_1,\ldots,k_i}c^j_{m_1,\ldots,m_t}c^{s-j}_{k_1,\ldots,k_i}\left[A_t\left(Y_{m_1}\odot\cdots\odot Y_{m_t}\right)\right]\odot Y_{k_1}\odot\cdots\odot Y_{k_i}.\tag{43}$$

The above multi-sums are indexed, respectively, by sets $J_{t,j}$ and $J_{i,s-j}$, where

$$J_{r,c}:=\left\{(n_1,\ldots,n_r)\in\mathbb{Z}^r:1\leqslant n_1\leqslant n_2\leqslant\cdots\leqslant n_r\text{ and }n_1+\cdots+n_r=c\right\}.$$

The set of all ordered concatenations of index vectors in $J_{t,j}$ and $J_{i,s-j}$ as j varies from t to $s-i$ equals the complete set $J_{t+i,s}$. Conversely, for every multi-index

$$\mathbf{n}=(n_1,\ldots,n_t,n_{t+1},\ldots,n_{t+i})\in J_{t+i,s},$$

consider the set of pairs of multi-indices $\mathbf{m}=(m_1,\ldots,m_t)$ and $\mathbf{k}=(k_1,\ldots,k_i)$ whose ordered concatenation is $\mathbf{n}$:

$$\mathcal{I}_{t,i}\left(\mathbf{n}\right)=\left\{(\mathbf{m},\mathbf{k}):\sigma\left(m_1,\ldots,m_t,k_1,\ldots,k_i\right)=\mathbf{n}\text{ for some }\sigma\in\mathfrak{S}_{i+t}\right\}$$

the terms in (43) indexed by $\mathcal{I}_{t,i}\left(\mathbf{n}\right)$ are summed up in

$$\sum_{(\mathbf{m},\mathbf{k})\in\mathcal{I}_{t,i}(\mathbf{n})}\binom{s}{|\mathbf{m}|}c_{\mathbf{m}}\,c_{\mathbf{k}}\left[A_t\left(Y_{m_1}\odot\cdots\odot Y_{m_t}\right)\right]\left(Y_{k_1}\odot\cdots\odot Y_{k_i}\right).$$

Let us discriminate among terms in the above sum. For every $(\mathbf{m},\mathbf{k})\in\mathcal{I}_{t,i}(\mathbf{n})$, split $\mathbf{m}$ and $\mathbf{k}$ into copies of different integers:

$$\left.\begin{aligned}\mathbf{m} &= \left(\mu_1 \overset{M_1}{\cdots} \mu_1,\cdots,\mu_p \overset{M_p}{\cdots} \mu_p\right)\\ \mathbf{k} &= \left(\mu_1 \overset{K_1}{\cdots} \mu_1,\cdots,\mu_q \overset{K_q}{\cdots} \mu_q\right)\end{aligned}\right\} \qquad 1\leqslant \mu_1<\mu_2<\cdots<\mu_{\max\{p,q\}}. \tag{44}$$

This obviously implies (equalling M_i or K_i multiplicities to zero whenever necessary)

$$\mathbf{n}=\left(\mu_1 \overset{M_1+K_1}{\cdots} \mu_1,\cdots,\mu_{\max\{p,q\}} \overset{M_{\max\{p,q\}}+K_{\max\{p,q\}}}{\cdots} \mu_{\max\{p,q\}}\right), \tag{45}$$

and

$$\sum_{\nu=1}^{\max\{p,q\}} M_\nu+K_\nu=t+i, \qquad \sum_{\nu=1}^{\max\{p,q\}} M_\nu\mu_\nu+K_\nu\mu_\nu=s;$$

the amount of permutations of $\mathbf{n}$ in (45) leaving $\mathbf{m}$ and $\mathbf{k}$ in (44) invariant is equal to

$$\binom{M_1+K_1}{M_1}\cdot\binom{M_2+K_2}{M_2}\cdots\binom{M_{\max\{p,q\}}+K_{\max\{p,q\}}}{M_{\max\{p,q\}}}.$$

Multiplication of this product by $c_{\mathbf{n}}$ yields (writing $r=\max\{p,q\}$)

$$\frac{\binom{M_1+K_1}{M_1}\cdot\binom{M_2+K_2}{M_2}\cdots\binom{M_r+K_r}{M_r}\cdot\binom{s}{\mathbf{n}}}{(M_1+K_1)!\,(M_2+K_2)!\cdots(M_r+K_r)!}=\frac{s!}{M_1!K_1!\mu_1!^{M_1+K_1}\cdots M_r!K_r!\mu_r!^{M_r+K_r}}. \tag{46}$$

Let us now return to sum (43). Using multiplicities as in (44), the summand corresponding to a given $(\mathbf{m},\mathbf{k})\in\mathcal{I}_{t,i}(\mathbf{n})$ has its coefficient equal to

$$\binom{s}{|\mathbf{m}|}c_{\mathbf{m}}\,c_{\mathbf{k}}=\binom{s}{M_1\mu_1+\cdots+M_p\mu_p}\frac{\binom{M_1\mu_1+\cdots+M_p\mu_p}{\mathbf{m}}}{M_1!\cdots M_p!}\frac{\binom{K_1\mu_1+\cdots+K_q\mu_q}{\mathbf{k}}}{K_1!\cdots K_q!}$$

which simplifies into (46).

Hence $\binom{s}{j}c^j_{\mathbf{m}}c^{s-j}_{\mathbf{k}}$ times $\left[A_t\left(Y_{m_1}\odot\cdots\odot Y_{m_t}\right)\right]\odot Y_{k_1}\odot\cdots\odot Y_{k_i}$ equals $c^s_{\mathbf{m},\mathbf{k}}$ times all permutations of the factors leaving these products invariant. This allows us to apply Lemma 2.14 to A_t and $Y_{\odot\mathbf{m}}\odot Y_{\odot\mathbf{k}}:=Y_{m_1}\odot\cdots\odot Y_{m_t}\odot Y_{k_1}\odot\cdots\odot Y_{k_i}$:

$$\binom{s}{j}c^j_{\mathbf{m}}c^{s-j}_{\mathbf{k}}\left[A_t\left(Y_{m_1}\odot\cdots\odot Y_{m_t}\right)\right]\odot Y_{k_1}\odot\cdots\odot Y_{k_i}=\binom{i+t}{i}\left(A_t\odot\mathrm{Id}_n^i\right)Y_{\odot\mathbf{m}}\odot Y_{\odot\mathbf{k}}. \tag{47}$$

The fact every summand in (43) fits the same profile as the left-hand side in (47) allows us to factor $\binom{i+t}{i}\left(A_t\odot\mathrm{Id}_n^i\right)$ out of the whole sum, namely $Z_{i+t,s}$.

e) Replacing each factor Y_{i_j} by QY_{i_j} in (40) and applying Lemma 2.10 we obtain $\exp_\odot QY=\left(\tilde{Z}_{r,k}\right)$ where

$$\tilde{Z}_{r,s}=\sum_{i_1+\cdots+i_r=s}Q^{\odot r}\odot c^s_{i_1,\ldots,i_r}Y_{i_1}\odot Y_{i_2}\odot\cdots\odot Y_{i_r}=Q^{\odot r}Z_{r,s},$$

hence matrix $\exp_\odot Y$ appears multiplied by $\mathrm{diag}\left(\cdots,Q^{\odot 2},Q^{\odot 1},1\right)=\exp_\odot Q$.

□

Lemma 3.11. *Let A and Y be as in Lemma 3.10. Then,*

$$\left(A\exp_\odot Y\right)\odot\exp_\odot Y=\left(A\odot\exp_\odot\mathrm{Id}_n\right)\exp_\odot Y. \tag{48}$$

Proof. Upon observation of (34), $B := A \odot \exp_{\odot} \mathrm{Id}_n \in \mathrm{Mat}(K)$ is defined recursively by

$$B_1 = A_1, \qquad B_k = \left(\begin{array}{c|c} \binom{k}{k-1} A_1 \odot \mathrm{Id}_n^{\odot k-1} & \\ \binom{k}{k-2} A_2 \odot \mathrm{Id}_n^{\odot k-2} & \\ \vdots & B_{k-1} \\ \binom{k}{0} A_k & \end{array} \right), \quad k \geqslant 1.$$

Let

$$\Phi_1 = Y_1, \qquad \Phi_k = \left(\begin{array}{c|c} Z_{k,k} & \\ Z_{k-1,k} & \\ \vdots & \Phi_{k-1} \\ Z_{1,k} & \end{array} \right), \quad k \geqslant 2, \tag{49}$$

be the matrix formed by the first k row and column blocks in $\exp_{\odot} Y$. Block row r of B is

$$B_r := \left(\binom{k}{r-1} A_{k+1-r} \odot \mathrm{Id}_n^{\odot r-1} \;\middle|\; \binom{k-1}{r-1} A_{k-r} \odot \mathrm{Id}_n^{\odot r-1} \;\middle|\; \cdots \;\middle|\; \binom{r}{r-1} A_1 \odot \mathrm{Id}_n^{\odot r-1} \;\middle|\; 0 \;\middle|\; \cdots \;\middle|\; 0 \right),$$

the row comprised of the first k blocks in A is written $A_k = (A_{1,k}, A_{1,k-1} \ldots, A_{1,1})$ and the first block column in Φ_k is $Z^k := (Z_{k,k}, Z_{k-1,k} \ldots, Z_{1,k})^T$.

For every $s \geqslant 1$, block ${}_{1,s}$ in $A \exp_{\odot} Y$ is equal to

$$X_s = A_s Z^s = \sum_{j=1}^{s} A_j Z_{j,s},$$

hence for every $r = 1, \ldots, k$ block ${}_{r,k}$ in $(A \exp_{\odot} Y) \odot \exp_{\odot} Y$ is equal to

$$\sum_{j=1}^{k-r+1} \binom{k}{j} X_j \odot Z_{r-1,k-j} = \sum_{j=1}^{k-r+1} \binom{k}{j} \left(\sum_{t=1}^{j} A_t Z_{t,j} \right) \odot Z_{r-1,k-j},$$

which can be rewritten as

$$\sum_{t=1}^{k-r+1} \sum_{j=t}^{k-r+1} \binom{k}{j} (A_t Z_{t,j}) \odot Z_{r-1,k-j},$$

its innermore sum ostensibly calling for Lemma 3.10 reenacted with $p = 1$, $s = k$ and $i = r - 1$. (41) indeed yields the above equal to

$$\sum_{t=1}^{k-r+1} \binom{t+r-1}{r-1} \left(A_t \odot \mathrm{Id}_n^{\odot r-1} \right) Z_{t+r-1,k},$$

precisely block combination $B_r Z^k$. □

3.2 Application to power series

Since polynomials and power series split into homogeneous components, Example 3.7(**3**) implies:

Lemma 3.12.

a) Let $F \in K[[\boldsymbol{x}]]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$, be a formal series. Then there exists a set of row blocks $M_F^{1,i} \in \mathrm{Mat}_{m,n}^{1,i}(K)$, $i \geqslant 0$ such that F admits the expression $F(\boldsymbol{x}) = M_F \exp_{\odot} X$, where

$$M_F := \left(\begin{array}{ccc|c} & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \hline \cdots & 0 & 0 & 0 \end{array} \right) \in \mathrm{Mat}^{1,n}(K), \qquad X := \left(\begin{array}{cc|c} \ddots & \vdots & \vdots \\ \cdots & 0 & 0 \\ \cdots & 0 & \boldsymbol{x} \\ \hline \cdots & 0 & 0 \end{array} \right).$$

b) If $F = F_1 \times \cdots \times F_m$ is a vector *power series, adequate $M_F^{1,i} \in \mathrm{Mat}_{m,n}^{1,i}(K)$ render*

$$\boxed{F(\boldsymbol{x}) = M_F \exp_\odot X} \qquad \text{where } M_F := \left(\begin{array}{ccc|c} & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 \\ \cdots & M_F^{1,2} & M_F^{1,1} & M_F^{1,0} \\ \hline \cdots & 0 & 0 & 0 \end{array}\right) \in \mathrm{Mat}^{m,n}.$$

Following Definition 3.5, we write $F(\boldsymbol{x}) = M_F \exp_\odot \boldsymbol{x}$ if it poses no clarity issue. □

From the above Lemma it follows that every formal power series can be expressed in the form $M_F \exp_\odot \boldsymbol{x}$, where abusing notation once again

$$M_F = J_F + M_F^{1,0} := \left(\begin{array}{ccc|c} \cdots & M_F^{1,2} & M_F^{1,1} & 0 \\ \hline \cdots & 0 & 0 & 0 \end{array}\right) + \left(\begin{array}{c|c} 0 & M_F^{1,0} \\ \hline 0 & 0 \end{array}\right). \tag{50}$$

In other words: M_F equals the sum of two matrices with easily computable $\odot$-exponentials: one following Example 3.7 (**3**) (same as $\boldsymbol{x}$) and one following pattern (37). This fact, Lemma 3.6, the fact $(\mathrm{Mat}(K), +, \odot)$ is an integral domain and the universal property of $\odot$ on finite products yield the following two results; see [8, 10] for a proof.

Lemma 3.13. *Given power series $F = (F_1, \ldots, F_m)$ and $G = (G_1, \ldots, G_p)$ in n and m indeterminates, respectively,*

a) If $n = m$, $M_{FG} = M_F \odot M_G$.

b) $\exp_\odot F(\boldsymbol{x}) = (\exp_\odot M_F)(\exp_\odot \boldsymbol{x})$.

c) $M_{G\circ F} = M_G \exp_\odot M_F$.

d) $\exp_\odot (M_G \exp_\odot M_F) = (\exp_\odot M_G)(\exp_\odot M_F)$. □

Corollary 3.14. *Let $F(\boldsymbol{x}) = (F_1, \ldots, F_p)(x_1, \ldots, x_n)$ be a vector power series, $\boldsymbol{y} = F(\boldsymbol{x})$ and*

$$\left.\begin{array}{rcl} \boldsymbol{X} & = & R_{x,X} \exp_\odot \boldsymbol{x} \in K^N, \\ \boldsymbol{Y} & = & S_{y,Y} \exp_\odot \boldsymbol{y} \in K^P, \end{array}\right\} \qquad R_{x,X} \in \mathrm{Mat}^{N,n}(K), \quad S_{y,Y} \in \mathrm{Mat}^{P,p}(K),$$

be independent and dependent variable changes, which we assume admit formal inverse changes

$$\left.\begin{array}{rcl} \boldsymbol{x} & = & R_{X,x} \exp_\odot \boldsymbol{X}, \\ \boldsymbol{y} & = & S_{Y,y} \exp_\odot \boldsymbol{Y}, \end{array}\right\} \qquad R_{X,x} \in \mathrm{Mat}^{n,N}(K), \quad S_{Y,y} \in \mathrm{Mat}^{p,P}(K).$$

Then, the expression of F in the new variables, written in that in those old, is

$$\boxed{M_{F,X,Y} = S_{y,Y}\left(\exp_\odot M_{F,x,y}\right) \exp_\odot R_{X,x}} \qquad \text{where } \boldsymbol{y} = F(\boldsymbol{x}) = M_{F,x,y} \exp_\odot \boldsymbol{x}. \quad \square \tag{51}$$

As was hinted at in [10, p. 5], the above result shows interesting light on the way finite-level transformations translate into transformations on $\mathrm{Mat}^{n,m}$. For a linear transformation of the independent variables $\boldsymbol{x} = B\boldsymbol{X}$, however, basic properties of $\exp_\odot$ are as useful as (51) in proving F admits the following expression in the new variable $\boldsymbol{X}$ (mind the effect of the first matrix, equal to zero save for block $_{1,1}$ which is equal to Id_n, on the second one):

$$F(\boldsymbol{X}) = \mathrm{Id}_n (\exp_\odot M_F)(\exp_\odot B)\,\boldsymbol{X} = \left(J_F + M_F^{0,0}\right)(\exp_\odot B) \exp_\odot \boldsymbol{X}. \tag{52}$$

This will be applied to first integrals of dynamical systems in Section 5.

4 Higher-order variational equations

4.1 Structure

Let us step back to what was said in Section 1.2. For each particular integral curve $\boldsymbol{\phi} = \{\boldsymbol{\phi}(t) : t \in T \subseteq \mathbb{P}^1_{\mathbb{C}}\}$ of a given complex autonomous dynamical system (DS), the **variational system** VE^k_ϕ for (DS) along $\boldsymbol{\phi}$ is satisfied by the partial derivatives $\frac{\partial^k}{\partial \boldsymbol{z}^k}\varphi(t, \boldsymbol{\phi}(t))$.

Case $k = 1$ being trivial as shown in (VE_ϕ), the situation of interest is $k > 1$. We will eschew formulations such as those in [24, eq (14)] in favour of a two-fold explicit expression: plain, sum-related expansion (63), and equally plain formulae (53), (64), (LVE_ϕ) and (VE^k_ϕ) using Linear Algebra to express multilinear maps.

Notation 4.1. *Define $\boldsymbol{\phi}(t)$ be a particular solution of* (DS)*, $K := \mathbb{C}(\boldsymbol{\phi})$, $A_i := X^{(i)}(\boldsymbol{\phi})$ and $Y_i := \mathrm{lex}\left(\frac{\partial^i}{\partial \boldsymbol{z}^i}\varphi(t, \boldsymbol{\phi})\right)$, and following Lemma 3.10, let*

$$\Phi_1 = Y_1, \qquad \Phi_k = \begin{pmatrix} Z_{k,k} & \\ Z_{k-1,k} & \boxed{\quad\Phi_{k-1}\quad} \\ \vdots & \\ Z_{1,k} & \end{pmatrix}, \quad k \geqslant 2, \tag{53}$$

be formed by the first k block rows and columns in $\Phi = \exp_\odot Y$. Define $A, Y \in \mathrm{Mat}(K)$ as in Lemma 3.10 with the above terms A_i, Y_i as blocks.

We also denote the canonical basis on K^n (meaning the set of columns of Id_n) by $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$.

Lemma 4.2. *In the hypotheses described in Notation 4.1, let $k \geqslant 1$. Then,*

$$Y_k = \sum_{m=1}^{n} \frac{\partial Y_{k-1}}{\partial z_m}\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right), \tag{54}$$

and for every $m = 1, \ldots, n$,

$$\frac{\partial}{\partial z_m} Y_k = Y_{k+1}\left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right), \tag{55}$$

$$\frac{\partial}{\partial z_m} Z_{r,k} = Z_{r,k+1}\left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right) - \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot r-1}\right) Z_{r-1,k}, \quad \textit{for every } r \leqslant k, \tag{56}$$

$$\frac{\partial}{\partial z_m} A_k = A_{k+1}\left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right). \tag{57}$$

Proof. (54) is an immediate consequence of Lemma 2.12 and equation (55) which we are now going to prove. We have, for every given ordered multi-index $\mathbf{i} = (i_1, \ldots, i_k)$,

$$\frac{\partial Y_k}{\partial z_m}\boldsymbol{e}_{i_1} \odot \cdots \odot \boldsymbol{e}_{i_k} = \frac{\partial}{\partial z_m}\frac{\partial^k \varphi}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} = Y_{k+1}\boldsymbol{e}_m \odot \boldsymbol{e}_{i_1} \odot \cdots \odot \boldsymbol{e}_{i_k}.$$

The right-hand side in (55) is equal to this expression, too, by simple application of the same principle as in (20) in order to obtain $\left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right)\boldsymbol{e}^{\odot \mathbf{i}} = \boldsymbol{e}_m \odot \boldsymbol{e}^{\odot \mathbf{i}}$. The effect of $\frac{\partial}{\partial \boldsymbol{z}}$ on A_j is clear as well: chain rule implies

$$\frac{\partial A_k}{\partial z_m}\boldsymbol{e}_{i_1}\odot\cdots\odot\boldsymbol{e}_{i_k} = \frac{\partial}{\partial z_m}\frac{\partial X}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}} = \sum_{r=1}^{n}\frac{\partial^{k+1}X}{\partial z_{i_1}\partial z_{i_2}\cdots\partial z_{i_k}\partial z_r}\frac{\partial\varphi_r}{\partial z_m} = \sum_{r=1}^{n} A_{k+1}\left(\boldsymbol{e}_r \odot \boldsymbol{e}^{\odot \mathbf{i}}\right)\frac{\partial \varphi_r}{\partial z_m},$$

which is equal, again using (20) in order to obtain $\frac{\partial\varphi_r}{\partial z_m}\boldsymbol{e}_r \odot \boldsymbol{e}^{\odot\mathbf{i}} = \left(\frac{\partial\varphi_r}{\partial z_m}\boldsymbol{e}_r \odot \mathrm{Id}_n^{\odot r}\right)\boldsymbol{e}^{\odot\mathbf{i}}$, to

$$A_{k+1}\sum_{r=1}^{n}\left(\boldsymbol{e}_r \odot \boldsymbol{e}^{\odot\mathbf{i}}\right)\frac{\partial\varphi_r}{\partial z_m} = A_{k+1}\sum_{r=1}^{n}\left(\frac{\partial\varphi_r}{\partial z_m}\boldsymbol{e}_r \odot \boldsymbol{e}^{\odot\mathbf{i}}\right) = A_{k+1}\sum_{r=1}^{n}\left(\frac{\partial\varphi_r}{\partial z_m}\boldsymbol{e}_r \odot \mathrm{Id}_n^{\odot r}\right)\boldsymbol{e}^{\odot\mathbf{i}},$$

hence to

$$A_{k+1}\left(\frac{\partial\varphi}{\partial z_m}\odot \mathrm{Id}_n^{\odot k}\right)\boldsymbol{e}^{\odot \mathbf{i}} = A_{k+1}\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot k}\right)\boldsymbol{e}^{\odot \mathbf{i}}.$$

Let us prove (57) by induction over k. Assume the equation holds for all values smaller than k. Derivation of (38) and use of (55) and Leibniz rule (30) yields

$$\frac{\partial}{\partial z_m}Z_{r,k} = S_1+S_2-S_3,$$

where

$$\begin{aligned}
S_1 &:= \frac{1}{r}\sum_{j=1}^{k-r+1}\binom{k}{j}\left[Y_{j+1}\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot j}\right)\right]\odot Z_{r-1,k-j},\\
S_2 &:= \frac{1}{r}\sum_{j=1}^{k-r+1}\binom{k}{j}Y_j\odot\left[Z_{r-1,k-j+1}\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot k-j}\right)\right],\\
S_3 &:= \frac{1}{r}\sum_{j=1}^{k-r+1}\binom{k}{j}Y_j\odot\left[\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right)Z_{r-2,k-j}\right].
\end{aligned}$$

Completion of S_3 with $j=k-r+2$ and application of (41) with $i=1$, $t=r-2$, $s=k$, $A_t = Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}$ and $p=r-1$ yields

$$S_3 = \frac{r-1}{r}\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-1}\right)Z_{r-1,k} - \frac{1}{r}\binom{k}{k-r+2}Y_{k-r+2}\odot\left[\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right)Z_{r-2,r-2}\right]. \tag{58}$$

The second term in the above expression can be written in the same manner as summands in S_2; indeed, using (26) and the fact $Z_{r-2,r-2}=Y_1^{\odot r-2}$,

$$\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right)Z_{r-2,r-2} = Z_{r-1,r-1}\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right),$$

hence the second term in (58) is

$$\frac{1}{r}\binom{k}{k-r+2}Y_{k-r+2}\odot\left[\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right)Z_{r-2,r-2}\right] = \frac{1}{r}\binom{k}{k-r+2}Y_{k-r+2}\odot\left[Z_{r-1,r-1}\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-2}\right)\right],$$

namely the additional summand for $j=k-r+2$ in S_2.

A missing term $\frac{1}{r}\left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-1}\right)Z_{r-1,k}$ needs to be accounted for in the first summand of (58). But it equals term $j=0$ for S_1. Hence, index shift puts S_1, S_2 and the two extra terms from (58) together in a single sum:

$$\frac{\partial Z_{r,k}}{\partial z_m} = \frac{1}{r}\sum_{j=1}^{k-r+2}s_j - \left(Y_1\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot r-1}\right)Z_{r-1,k},$$

where

$$s_j = \binom{k}{j-1}\left[Y_j\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot j-1}\right)\right]\odot Z_{r-1,k-j+1} + \binom{k}{j}Y_j\odot\left[Z_{r-1,k-j+1}\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot k-j}\right)\right]. \tag{59}$$

Let us check s_j is equal to $\binom{k+1}{j}\left(Y_j\odot Z_{r-1,k+1-j}\right)\left(\boldsymbol{e}_m\odot \mathrm{Id}_n^{\odot k}\right)$. The columns of the latter matrix are of the following form, defining $\mathbf{1}_m = \left(0,\ldots,\overset{(m)}{1},\ldots,0\right)\in\mathbb{Z}^n$ and whenever $|\mathbf{k}|=k$:

$$\binom{k+1}{j}\left(Y_j\odot Z_{r-1,k+1-j}\right)\boldsymbol{e}^{\odot \mathbf{k}+\mathbf{1}_m} = \sum_{|\mathbf{i}|=j}\binom{\mathbf{k}+\mathbf{1}_m}{\mathbf{i}}Y_j\boldsymbol{e}^{\odot\mathbf{i}}\odot Z_{r-1,k-j+1}\boldsymbol{e}^{\odot \mathbf{k}+\mathbf{1}_m-\mathbf{i}},$$

and can be split into two sums depending on whether $\mathbf{i} = (i_1, \ldots, i_n)$ above satisfies $i_m > 0$:

$$\sum_{|\mathbf{p}|=j-1} \binom{\mathbf{k}}{\mathbf{p}} Y_j \left(\boldsymbol{e}_m \odot \boldsymbol{e}^{\odot \mathbf{p}}\right) \odot Z_{r-1,k-j+1} \boldsymbol{e}^{\odot \mathbf{k}-\mathbf{p}} + \sum_{|\mathbf{q}|=j} \binom{\mathbf{k}}{\mathbf{q}} Y_j \boldsymbol{e}^{\odot \mathbf{q}} \odot Z_{r-1,k-j+1} \left(\boldsymbol{e}_m \odot \boldsymbol{e}^{\odot \mathbf{k}-\mathbf{q}}\right),$$

precisely $s_j \boldsymbol{e}^{\mathbf{k}}$ as in (59). Hence

$$\begin{aligned} \frac{\partial Z_{r,k}}{\partial z_m} &= \frac{1}{r} \sum_{j=1}^{k-r+2} \binom{k+1}{j} Y_j \odot Z_{r-1,k+1-j} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right) - \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot r-1}\right) Z_{r-1,k} \\ &= Z_{r,k+1} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k}\right) - \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot r-1}\right) Z_{r-1,k}. \end{aligned}$$

□

Proposition 4.3 (First explicit version of non-linearised VE_ϕ^k)**.** *In the above hypotheses,*

$$\dot{Y} = A \exp_\odot Y; \tag{VE_ϕ}$$

in other words, for every $k \geqslant 1$,

$$\frac{d}{dt} Y_k = \sum_{j=1}^{k} A_j Z_{k,j} = \sum_{j=1}^{k} A_j \sum_{i_1+\cdots+i_j=k} c^k_{i_1,\ldots,i_j} Y_{i_1} \odot Y_{i_2} \odot \cdots \odot Y_{i_j}. \tag{VE_ϕ^k}$$

Proof. Assume the result is true for $k-1$, and let us prove it for k. That is, assume VE_ϕ^{k-1} can be expressed in the form $\frac{d}{dt} Y_{k-1} = \sum_{j=1}^{k-1} A_j Z_{j,k-1}$. We recall the entries in Y_{k-1} are partial derivatives of $\varphi(t, \boldsymbol{z})$, hence $\frac{d}{dt} \equiv \frac{\partial}{\partial t}$ on every entry, Schwarz Lemma applies and derivation of (54) yields

$$\frac{d}{dt} Y_k = \sum_{m=1}^{n} \frac{\partial}{\partial t} \frac{\partial Y_{k-1}}{\partial z_m} \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \sum_{m=1}^{n} \frac{\partial}{\partial z_m} \frac{\partial Y_{k-1}}{\partial t} \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right);$$

induction hypothesis and Leibniz rule render $\frac{d}{dt} Y_k$ equal to

$$\sum_{m=1}^{n} \frac{\partial}{\partial z_m} \left[\sum_{p=1}^{k-1} A_p Z_{p,k-1}\right] \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \sum_{m=1}^{n} \left[\sum_{p=1}^{k-1} \frac{\partial A_p}{\partial z_m} Z_{p,k-1} + A_p \frac{\partial Z_{p,k-1}}{\partial z_m}\right] \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right);$$

equations (56) and (57) imply this is equal to $S_1 + S_2 - S_3$, where

$$\begin{aligned} S_1 &= \sum_{m=1}^{n} \sum_{p=1}^{k-1} A_{p+1} \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot p}\right) Z_{p,k-1} \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right), \\ S_2 &= \sum_{m=1}^{n} \sum_{p=1}^{k-1} A_p Z_{p,k} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right) \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right), \\ S_3 &= \sum_{m=1}^{n} \sum_{p=1}^{k-1} A_p \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot p-1}\right) Z_{p-1,k-1} \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right). \end{aligned}$$

Sum swapping in $\sum_m \sum_p$ implies

$$S_2 = \sum_{p=1}^{k-1} A_p Z_{p,k} \sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right) \left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = \sum_{p=1}^{k-1} A_p Z_{p,k}. \tag{60}$$

Simple index shift on p and (39) render $S_1 - S_3$ equal to

$$\sum_{m=1}^{n} A_k \left(Y_1 \boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right) Z_{k-1,k-1}\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = A_k \sum_{m=1}^{n} \left(Y_1 \odot Z_{k-1,k-1}\right)\left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right)\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right),$$

which is equal to

$$A_k \left(Y_1 \odot Z_{k-1,k-1}\right) \sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k-1}\right)\left(\boldsymbol{e}_m^T \odot \mathrm{Id}_n^{\odot k-1}\right) = A_k \left(Y_1 \odot Z_{k-1,k-1}\right) = A_k Z_{k,k},$$

the missing summand in (60). □

The following is but a reformulation of the above result:

Corollary 4.4 (Second explicit version of non-linearised VE_ϕ^k)**.** *Let $\varphi(t,\boldsymbol{\phi}) = (\phi_1, \ldots, \phi_n)$ denote the flow of (DS). Let $k \geqslant 1$ be the order of the variational system. Given integers $N_1, \ldots, N_k \geqslant 1$, $r = 1, \ldots, k$ and $m_1, m_2, \ldots, m_r \geqslant 0$ such that $\sum_{j=1}^{r} m_j = k$, define:*

a) $S_{N_1,\ldots,N_k} := \{\sigma(N_1, \ldots, N_k) : \sigma \in \mathfrak{S}_k\}$ and the set of partitions of $\{N_1, \ldots, N_k\}$ in ordered subsets of sizes $m_1, \ldots, m_r$:

$$I_{N_1,\ldots,N_k}^{m_1,\ldots,m_r} := \{(\mathbf{K}_1, \ldots, \mathbf{K}_r) \in S_{N_1,\ldots,N_k} : \mathbf{K}_i = (K_{i,1}, \ldots, K_{i,m_i}),\ K_{i,1} < \cdots < K_{i,m_i}\}; \tag{61}$$

b) and, using abridged notation $\sum_{j_1,\ldots,j_r}$ to denote $\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\cdots\sum_{j_r=1}^{n}$,

$$T_{N_1,\ldots,N_k}^{m_1,\ldots,m_r} := \sum_{(\mathbf{K}_1,\ldots,\mathbf{K}_r)\in I_{N_1,\ldots,N_k}^{m_1,\ldots,m_r}} \sum_{j_1,\ldots,j_r} \frac{\partial^r X_i}{\partial z_{j_1}\cdots\partial z_{j_r}} \frac{\partial^{m_1}\phi_{j_1}}{\partial \boldsymbol{z}_{\mathbf{K}_1}} \cdots \frac{\partial^{m_r}\phi_{j_r}}{\partial \boldsymbol{z}_{\mathbf{K}_r}}. \tag{62}$$

Then, the order-k variational equation along $\boldsymbol{\phi} = \{\boldsymbol{\phi}(t)\}$ is summarised in the following:

$$\frac{d}{dt}\frac{\partial^k \phi_i}{\partial z_{N_1}\partial z_{N_2}\cdots\partial z_{N_k}} = \sum_{r=1}^{k}\sum_{m_1,\ldots,m_r} T_{N_1,\ldots,N_k}^{m_1,\ldots,m_r}, \qquad i, N_1, \ldots, N_k \in \{1, \ldots, n\}, \tag{63}$$

indices in $\sum_{m_1,\ldots,m_r}$ again constrained by $0 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_r$ and $m_1 + \cdots + m_r = k$. □

In the previous Lemma we effectively settled the entries for lower n rows in $A_{\mathrm{LVE}_\phi^k}$ and the first n columns in Φ_k by virtue of (VE_ϕ^k). Let us now prove the result true for the rest of the matrices.

Proposition 4.5 (Explicit version of LVE_ϕ^k)**.** *Still following Notation 4.1, the infinite system*

$$\boxed{\dot{X} = A_{\mathrm{LVE}_\phi} X,} \qquad A_{\mathrm{LVE}_\phi} := A \odot \exp_\odot \mathrm{Id}_n, \tag{LVE_ϕ}$$

has $\Phi := \exp_\odot Y$ as a solution matrix. Hence, for every $k \geqslant 1$,

a) the lower-triangular recursive $D_{n,k} \times D_{n,k}$ form for LVE_ϕ^k is $\dot{Y} = A_{\mathrm{LVE}_\phi^k} Y$, its system matrix being obtained from the first k row and column blocks of A_{LVE_ϕ}:

$$A_{\mathrm{LVE}_\phi^k} = \left(\begin{array}{cc} \binom{k}{k-1} A_1 \odot \mathrm{Id}_n^{\odot k-1} & \\ \binom{k}{k-2} A_2 \odot \mathrm{Id}_n^{\odot k-2} & \\ \vdots & \boxed{A_{\mathrm{LVE}_\phi^{k-1}}} \\ \binom{k}{0} A_k & \end{array}\right), \tag{64}$$

b) and the principal fundamental matrix for LVE_ϕ^k *is* Φ_k *from* $\Phi = \exp_\odot Y$ *in Notation 4.1.*

Proof. (48) in 3.11, (VE_ϕ) in Proposition 4.3, and item (b) in Lemma 3.6 imply

$$\overbrace{\exp_\odot Y}^{\cdot} = \dot{Y} \odot \exp_\odot Y = \left(A \exp_\odot Y\right) \odot \exp_\odot Y = \left(A \odot \exp_\odot \mathrm{Id}_n\right) \exp_\odot Y.$$

The rest follows from Lemma 3.10. ◻

Example 4.6. For instance, for $k = 5$ we have

$$A_{\mathrm{LVE}_\phi^5} = \begin{pmatrix} 5A_1 \odot \mathrm{Id}_n^{\odot 4} & & & & \\ 10A_2 \odot \mathrm{Id}_n^{\odot 3} & 4A_1 \odot \mathrm{Id}_n^{\odot 3} & & & \\ 10A_3 \odot \mathrm{Id}_n^{\odot 2} & 6A_2 \odot \mathrm{Id}_n^{\odot 2} & 3A_1 \odot \mathrm{Id}_n^{\odot 2} & & \\ 5A_4 \odot \mathrm{Id}_n & 4A_3 \odot \mathrm{Id}_n & 3A_2 \odot \mathrm{Id}_n & 2A_1 \odot \mathrm{Id}_n & \\ A_5 & A_4 & A_3 & A_2 & A_1 \end{pmatrix}$$

and, using any of the equivalent expressions (38), (40), the fundamental matrix having $\mathrm{Id}_{D_{n,5}}$ as an initial condition is

$$\Phi_5 = \begin{pmatrix} Y_1^{\odot 5} & & & & \\ 10Y_1^{\odot 3} \odot Y_2 & Y_1^{\odot 4} & & & \\ 10Y_1^{\odot 2} \odot Y_3 + 15Y_1 \odot Y_2^{\odot 2} & 6Y_1^{\odot 2} \odot Y_2 & Y_1^{\odot 3} & & \\ 10Y_2 \odot Y_3 + 5Y_1 \odot Y_4 & 4Y_1 \odot Y_3 + 3Y_2 \odot Y_2 & 3Y_1 \odot Y_2 & Y_1^{\odot 2} & \\ Y_5 & Y_4 & Y_3 & Y_2 & Y_1 \end{pmatrix},$$

hence (VE_ϕ^k) for $k = 5$ can be expressed as

$$\dot{Y}_5 = A_1 Y_5 + A_2 \left(10Y_2 \odot Y_3 + 5Y_1 \odot Y_4\right) + A_3 \left(10Y_1^{\odot 2} \odot Y_3 + 15Y_1 \odot Y_2^{\odot 2}\right) + A_4 \left(10Y_1^{\odot 3} \odot Y_2\right) + A_5 Y_1^{\odot 5}.$$

4.2 Explicit solution and monodromy matrices for LVE_ϕ^k

Let $T \subseteq \mathbb{P}^1_{\mathbb{C}}$ be the domain for time variable t in (DS) and $\gamma \subset T$ a closed path based at point $t_0 \in T$. Analytic continuation extends to polynomial functions, hence to symmetric products as seen in (16). Assume $k = 1$. If Y_1 is a fundamental matrix of first-order (VE_ϕ), analytic continuation along γ yields

$$Y_1(t_0) \xrightarrow[\text{cont}]{\gamma} Y_1(t_0) \cdot M_{1,\gamma},$$

$M_{1,\gamma}$ being the *monodromy matrix* ([31]) of (VE_ϕ). Assume $Y_1 := \Phi_1$ is the principal fundamental matrix for (VE_ϕ), any other fundamental matrix Ψ_1 recovered from $\Psi_1 = Y_1 \Psi_1(t_0)$.

Having computed Y_1, the non-linearised second-order equation, after Proposition 4.3, is

$$\dot{Y}_2 = A_1 Y_2 + A_2 \cdot \mathrm{Sym}^2(Y_1). \tag{VE_ϕ^2}$$

Following Proposition 4.5, linearised completion LVE_ϕ^2 has principal fundamental matrix

$$\Phi_2 = \begin{pmatrix} Y_1^{\odot 2} & \\ Y_2 & Y_1 \end{pmatrix}.$$

A particular solution Y_2 of (VE_ϕ^2) is found via usual variation of constants:

$$Y_2 = Y_1 \int Y_1^{-1} A_2 \mathrm{Sym}^2(Y_1),$$

which becomes a contour integral whenever time is taken along path γ:

$$Y_2 \xrightarrow[\text{cont}]{\gamma} Q_{1,2,\gamma} := M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 \mathrm{Sym}^2 \left(Y_1\right), \tag{65}$$

hence

$$\mathrm{Id}_n = \Phi_2 \left(t_0\right) \xrightarrow[\text{cont}]{\gamma} \begin{pmatrix} Y_1^{\odot 2}\left(t_0\right) M_{1,\gamma}^{\odot 2} & 0 \\ Y_1 \left(t_0\right) Q_{1,2,\gamma} & Y_1 \left(t_0\right) M_{1,\gamma} \end{pmatrix} = \Phi_2 \left(t_0\right) \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix},$$

and $[\gamma] \mapsto M_{i,\gamma}$ is a group morphism $\pi_1 \left(T, t_0\right) \to \mathrm{GL}_{D_{n,i}}\left(\mathbb{C}\right)$, hence for *any* fundamental matrix

$$\Psi_2 \left(t_0\right) \xrightarrow[\text{cont}]{\gamma} \Psi_2 \left(t_0\right) \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix};$$

therefore the monodromy of LVE_ϕ^2 along γ will be

$$M_{2,\gamma} := \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix} = \begin{pmatrix} M_{1,\gamma}^{\odot 2} & 0 \\ M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 Y_1^{\odot 2} & M_{1,\gamma} \end{pmatrix} \tag{66}$$

Assume $k = 3$. The principal fundamental matrix of LVE_ϕ^3 will be

$$\Phi_3 = \begin{pmatrix} \mathrm{Sym}^3 \left(Y_1\right) & & \\ 3Y_1 \odot Y_2 & \mathrm{Sym}^2 \left(Y_1\right) & \\ Y_3 & Y_2 & Y_1 \end{pmatrix}$$

and any other fundamental matrix can be expressed in the form $\Psi_3 = \Phi_3 C$ as usual. Let us now find a solution to

$$\dot{Y_3} = A_1 Y_3 + 3A_2 Y_1 \odot Y_2 + A_3 \mathrm{Sym}^3 \left(Y_1\right), \tag{67}$$

Same as before, a particular solution Y_3 of (67) is $Y_3 = Y_1 V_3$ where

$$\dot{V_3} = Y_1^{-1} \left(3A_2 Y_1 \odot Y_2 + A_3 \mathrm{Sym}^3 \left(Y_1\right)\right),$$

yielding a new contour integral if $\tau \in \gamma$:

$$Y_3 \xrightarrow[\text{cont}]{\gamma} Q_{1,3,\gamma} := M_{1,\gamma} \int_\gamma Y_1^{-1} \left(3A_2 Y_1 \odot Y_2 + A_3 \mathrm{Sym}^3 \left(Y_1\right)\right) d\tau. \tag{68}$$

The remaining term of our monodromy matrix is a direct consequence of analytic continuation as performed on $3Y_1 \odot Y_2$:

$$0 = 3Y_1 \left(t_0\right) \odot Y_2 \left(t_0\right) \xrightarrow[\text{cont}]{\gamma} 3M_{1,\gamma} \odot Q_{1,2,\gamma} = 3M_{1,\gamma} \odot \left(M_{1,\gamma} \int_\gamma Y_1^{-1} A_2 Y_1^{\odot 2}\right)$$

Our monodromy matrix is

$$M_{3,\gamma} := \begin{pmatrix} M_{1,\gamma}^{\odot 3} & & \\ 3M_{1,\gamma} \odot Q_{1,2,\gamma} & M_{1,\gamma}^{\odot 2} & \\ Q_{1,3,\gamma} & Q_{1,2,\gamma} & M_{1,\gamma} \end{pmatrix} = \begin{pmatrix} M_{1,\gamma}^{\odot 3} & \\ 3M_{1,\gamma} \odot Q_{1,2,\gamma} & \boxed{M_{2,\gamma}} \\ Q_{1,3,\gamma} & \end{pmatrix} \tag{69}$$

The pattern is clear now. Assume we have computed solutions $Y_1, \ldots, Y_{k-1}$ and performed continuation up to $k-1$:

$$\Phi_{k-1} \xrightarrow[\text{cont}]{\gamma} \Phi_{k-1} M_{k-1,\gamma} := \Phi_{k-1} \begin{pmatrix} Q_{k-1,k-1,\gamma} & & & & \\ Q_{k-2,k-1,\gamma} & Q_{k-2,k-2,\gamma} & & & \\ \vdots & \vdots & \ddots & & \\ Q_{2,k-1,\gamma} & Q_{2,k-2,\gamma} & \cdots & Q_{2,2,\gamma} & \\ Q_{1,k-1,\gamma} & Q_{1,k-2,\gamma} & \cdots & Q_{1,2,\gamma} & Q_{1,1,\gamma} \end{pmatrix},$$

where

$$Q_{r,s,\gamma} := \sum_{i_1+\cdots+i_r=s} c^s_{i_1,\dots,i_r} Q_{1,i_1,\gamma} \odot Q_{1,i_2,\gamma} \odot \cdots \odot Q_{1,i_r,\gamma}, \qquad s \geqslant r \geqslant 2. \tag{70}$$

Then, the fundamental matrix for $\left(\mathrm{LVE}^k_\phi\right)$ will be expressed in the form (53), its lower left block Y_k being computable in terms of the blocks $Z_{2,k},\dots,Z_{k,k}$ above it (all of which involve $Y_1,\dots,Y_{k-1}$) in virtue of (VE^k_ϕ): $Y_k = Y_1 V_k$, which is continued into $Q_{1,k,\gamma} := M_{1,\gamma}\int_\gamma V_k$, where $\dot{\overline{V_k}} = Y_1^{-1}\sum_{j=2}^k A_j Z_{j,k}$. Upper terms $Z_{2,k},\dots,Z_{k,k}$ are continued into $Q_{2,k},\dots,Q_{k,k}$ as in (70), s replaced by k. It is clear we have proven the following:

Lemma 4.7. *The monodromy matrix Φ_k of LVE^k_ϕ along closed path γ is composed by the first k row and column blocks in*

$$\exp_\odot Q_\gamma := \exp_\odot \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & Q_{1,2,\gamma} & Q_{1,1,\gamma} & 0 \\ \hline \cdots & 0 & 0 & 0 \end{array}\right), \tag{71}$$

where $Q_{1,1,\gamma} := M_{1,\gamma}$, blocks above the bottom row are computed according to (70) and

$$Q_{1,s,\gamma} := M_{1,\gamma}\int_\gamma Y_1^{-1}\sum_{j=2}^{s} A_j Z_{j,s}, \quad 2 \leqslant s \leqslant k. \qquad \square \tag{72}$$

Hence it is clear the computation of a monodromy matrix follows a block order such as the one below, blocks in the bottom row requiring quadratures:

$$\begin{array}{ccccc} \ddots & & & & \\ \cdots & \boxed{7} & & & \\ \cdots & \boxed{8} & \boxed{4} & & \\ \cdots & \boxed{9} & \boxed{5} & \boxed{2} & \\ \cdots & \boxed{10} & \boxed{6} & \boxed{3} & \boxed{1} \end{array} \tag{73}$$

Computing the monodromy matrix is concomitant to computing the fundamental matrix, i.e. said bottom-row quadratures must be both indefinite (yielding terms $Z_{1,s}$ to be used in the computation of $Z_{j,s}$ in (72)) and contour integrals. See Section 6 for an example.

We assume there are two generators $[\gamma],[\tilde{\gamma}] \in \pi_1(T;t_0)$, yielding two different matrices:

$$\gamma \longleftrightarrow Q_\gamma, \qquad \tilde{\gamma} \longleftrightarrow Q_{\tilde{\gamma}}$$

Commutativity of monodromy matrices now admits simple, compact formulation:

Proposition 4.8. *Two monodromy matrices $M_{k,\gamma}$ and $M_{k,\tilde{\gamma}}$ for LVE^k_ϕ commute if, and only if, their previous blocks $M_{k-1,\gamma}, M_{k-1,\tilde{\gamma}}$ commute and the additional properties hold*

$$\sum_{j=r}^{k} Q_{r,j,\gamma} Q_{j,k,\tilde{\gamma}} = \sum_{j=r}^{k} Q_{r,j,\tilde{\gamma}} Q_{j,k,\gamma}, \qquad \text{for every } r = 1,\dots,k-1,$$

matrices defined as in (70) and (72). $\square$

Remarks 4.9.

a) The monodromy group of a linear system is contained in its differential Galois group (e.g. [28]). The motivation for the above Lemma and Proposition is to capitalise on this fact. This may in turn be a step towards future constructive incarnations of the Morales-Ramis-Simó Theorem 1.3. The main obstacle implementing Proposition 4.8, symbolico-computational issues aside, is the incertitude on whether $M_{k,\gamma}$ and $M_{k,\tilde{\gamma}}$ belong to the Zariski identity component $\mathrm{Gal}\left(\mathrm{LVE}^k_\phi\right)^\circ$; a sufficient condition for arbitrary order is fulfilment at order 1, $M_{1,\gamma}, M_{1,\tilde{\gamma}} \in \mathrm{Gal}\left(\mathrm{VE}_\phi\right)^\circ$, itself an open problem in general.

b) All disquisitions and results on the variational jet in [20, 21] are referred to the lower n-row strip for commutators of these monodromies. More specifically:

- what is called *jet* therein is lower strip Y in principal fundamental matrix $\Phi = \exp_\odot Y$ for infinite system (LVE$_\phi$), and we will use this terminology in the following Section;
- morphism properties imply monodromy matrices along path commutators are equal to monodromy matrix commutators: $M_{k,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1} = M_{k,\gamma_2}^{-1} M_{k,\gamma_1}^{-1} M_{k,\gamma_2} M_{k,\gamma_1}$;
- hence, the "jet commutation" properties in [20, 21] amount to lower strip $Q_{k,\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1}$ (that is Y after passage along $\gamma_2^{-1}\gamma_1^{-1}\gamma_2\gamma_1$) equalling Id_n.

Although [20, 21] clearly benefit from the use of automatic differentiation techniques (see also [19]), it may be argued that expressions such as those in (LVE$_\phi$) provide for a fuller control of the general structure of the whole variational complex when it comes to symbolic computations, as well as a further check aid for the aforementioned techniques. See Subsection 6.1 for an example.

5 First integrals and higher-order variational equations

Let $F : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$ be a holomorphic function and $\boldsymbol{\phi} : I \subset \mathbb{C} \to U$. Firstly, the flow $\varphi(t, \boldsymbol{z})$ of X admits, at least formally, Taylor expansion (1) along $\boldsymbol{\phi}$ which is expressible as

$$\varphi(t, \boldsymbol{\phi} + \boldsymbol{\xi}) = \boldsymbol{\phi} + Y_1\boldsymbol{\xi} + \frac{1}{2}Y_2\boldsymbol{\xi}^{\odot 2} + \cdots = \boldsymbol{\phi} + J_\phi \exp_\odot \boldsymbol{\xi}, \tag{74}$$

where J_ϕ is the jet for flow $\varphi(t,\cdot)$ along $\boldsymbol{\phi}$, displayed as Y in (37) and defined in Notation 4.1 – that is, the matrix whose $\odot$-exponential Φ is a solution matrix for (LVE$_\phi$).

Secondly, the Taylor expansion of F along $\boldsymbol{\phi}$ can be written, cfr. [5, Lemma 2] and Notation 1.4, as

$$F(\boldsymbol{y} + \boldsymbol{\phi}) = F(\boldsymbol{\phi}) + \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle F^{(m)}(\boldsymbol{\phi}), \mathrm{Sym}^m \boldsymbol{y} \right\rangle. \tag{75}$$

Basic scrutiny of Example 3.7(**3**), Lemma 3.12 and (50) trivially implies (75) can be expressed as $F(\boldsymbol{y} + \boldsymbol{\phi}) = M_F^\phi \exp_\odot \boldsymbol{y}$, where

$$M_F^\phi = J_F^\phi + F^{(0)}(\boldsymbol{\phi}) := \left(\begin{array}{ccc|c} \cdots & 0 & 0 & 0 \\ \cdots & F^{(2)}(\boldsymbol{\phi}) & F^{(1)}(\boldsymbol{\phi}) & F^{(0)}(\boldsymbol{\phi}) \\ \hline \cdots & 0 & 0 & 0 \end{array}\right) \in \mathrm{Mat}^{1,n}(K),$$

i.e. J_F^ϕ is the jet or horizontal strip of lex-sifted partial derivatives of F at $\boldsymbol{\phi}$.

Definition 5.1. *We call*

$$\boxed{\dot{X} = A_{\mathrm{LVE}_\phi^\star} X,} \qquad A_{\mathrm{LVE}_\phi^\star} := -\left(A \odot \exp_\odot \mathrm{Id}_n\right)^T, \tag{LVE$_\phi^\star$}$$

*the **adjoint** or **dual** variational system of* (DS) *along ϕ. Same as in* (LVE$_\phi$) *and all throughout 4.1, consideration of finite subsystems, namely the lowest $D_{n,k} \times D_{n,k}$ block, leads to specific notation $\left(\mathrm{LVE}_\phi^k\right)^\star$.*

The following is well-known for finite systems and immediate upon derivation of equation $\Phi_k \Phi_k^{-1} = \mathrm{Id}_{D_{n,k}}$:

Lemma 5.2. $\left(\Phi_k^{-1}\right)^T$ *is a principal fundamental matrix of* $\left(\mathrm{LVE}_\phi^k\right)$ *for every* $k \geqslant 1$.

Hence, $\left(\Phi^{-1}\right)^T$*, is a solution to* ($\mathrm{LVE}_\phi^\star$)*, where* $\Phi = \exp_\odot J_\phi$. □

The following was proven in [24] and recounted in [5, Lemma 7], and may now be expressed in a simple, compact fashion:

Lemma 5.3. *Let* F *and* $\boldsymbol{\phi}$ *be a holomorphic first integral and a non-constant solution of* (DS) *respectively. Let* $V := J_F^T$ *be the transposed jet of* F *along* $\boldsymbol{\phi}$*. Then,* V *is a solution of* ($\mathrm{LVE}_\phi^\star$).

Proof. Let us recall formal expansion (74) and $F(\boldsymbol{y}) = J_F^\phi \exp_\odot \boldsymbol{y}$ for every $\boldsymbol{y} \in K^n$. Let $\boldsymbol{\psi} = \varphi(t, \boldsymbol{\phi} + \boldsymbol{\xi})$. We have, using Lemma 3.13,

$$F(\boldsymbol{\psi}) = F(\boldsymbol{\phi} + J_\phi \exp_\odot \boldsymbol{\xi}) = M_F^\phi \exp_\odot (J_\phi \exp_\odot \boldsymbol{\xi}) = \left(M_F^\phi \exp_\odot J_\phi\right) \exp_\odot \boldsymbol{\xi},$$

and $F(\boldsymbol{\psi})$ is supposed to be constant, hence applying (LVE_ϕ) and Lemma 3.13

$$0 = \dot{\overline{\left(M_F^\phi \exp_\odot J_\phi\right)}} \exp_\odot \boldsymbol{\xi} = \left(\dot{\overline{M_F^\phi}} + M_F^\phi A_{\mathrm{LVE}_\phi}\right) \exp_\odot J_\phi \exp_\odot \boldsymbol{\xi} = \left(\dot{\overline{M_F^\phi}} + M_F^\phi A_{\mathrm{LVE}_\phi}\right) \exp_\odot (\boldsymbol{\psi} - \boldsymbol{\phi}),$$

hence $\dot{\overline{M_F^\phi}} + M_F^\phi A_{\mathrm{LVE}_\phi} = 0$ leading us to the final result after transposing both sides. □

Compound the jet of field X, i.e. A in Notation 4.1 and Proposition 4.5, with a ${}_{1,0}$ term A_0, equal to $X^{(0)} = X(\boldsymbol{\phi}) = \dot{\boldsymbol{\phi}}$:

$$\widehat{A} := \left(\begin{array}{cccc|c} \cdots & 0 & 0 & & 0 \\ \cdots & A_2 & A_1 & & A_0 \\ \hline \cdots & 0 & 0 & & 0 \end{array}\right), \qquad A_i := X^{(i)}(\boldsymbol{\phi}) \in \mathrm{Mat}_n^{1,i}(K).$$

It is easy to check, via possibilities offered on i_1 and j_1 in (34), that the symmetric product of $\widehat{A}$ with $\exp_\odot \mathrm{Id}_n$ adds only a relatively minor addendum to A_{LVE_ϕ}, namely a superdiagonal of blocks $\binom{i}{i} A_0 \odot \mathrm{Id}_n^{\odot i} \in \mathrm{Mat}_n^{i+1,i}$, $i \geqslant 1$, effectively rendering it block-Hessenberg:

$$\widehat{A}_{\mathrm{LVE}_\phi} := \widehat{A} \odot \exp_\odot \mathrm{Id}_n = \lim_k \widehat{A}_{\mathrm{LVE}_\phi^k},$$

where, isolating $A_{\mathrm{LVE}_\phi^k}$ within $\widehat{A}_{\mathrm{LVE}_\phi^k}$ by means of a solid line,

$$\widehat{A}_{\mathrm{LVE}_\phi^k} := \left(\begin{array}{cccccc} A_0 \odot \mathrm{Id}_n^{\odot k} & & & & & \\ \binom{k}{k-1} A_1 \odot \mathrm{Id}_n^{\odot k-1} & A_0 \odot \mathrm{Id}_n^{\odot k-1} & & & & \\ \vdots & \vdots & \ddots & & & \\ \binom{k}{2} A_{k-2} \odot \mathrm{Id}_n^{\odot 2} & \binom{k-1}{2} A_{k-3} \odot \mathrm{Id}_n^{\odot 2} & \cdots & A_0 \odot \mathrm{Id}_n^{\odot 2} & & \\ \binom{k}{1} A_{k-1} \odot \mathrm{Id}_n & \binom{k-1}{1} A_{k-2} \odot \mathrm{Id}_n & \cdots & A_1 \odot \mathrm{Id}_n & A_0 \odot \mathrm{Id}_n & \\ A_k & A_{k-1} & \cdots & A_2 & A_1 & A_0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right) \tag{76}$$

$$= \left(\begin{array}{|c|c|} \hline \binom{k}{k} X^{(0)}(\boldsymbol{\phi}) \odot \mathrm{Id}_n^{\odot k} & \\ \hline \binom{k}{k-1} X^{(1)}(\boldsymbol{\phi}) \odot \mathrm{Id}_n^{\odot k-1} & \\ \binom{k}{k-2} X^{(2)}(\boldsymbol{\phi}) \odot \mathrm{Id}_n^{\odot k-2} & \widehat{A}_{\mathrm{LVE}_\phi^{k-1}} \\ \vdots & \\ \binom{k}{0} X^{(k)}(\boldsymbol{\phi}) \odot \mathrm{Id}_n^{\odot 0} & \\ \hline \end{array}\right).$$

Using the M_k–$\mathcal{M}_k$ notation in [5], it is immediate to check that

$$M_k^T = \mathrm{Id}_n^{\odot k-1} \odot \dot{\boldsymbol{\phi}} = \mathrm{Id}_n^{\odot k-1} \odot X(\boldsymbol{\phi}), \tag{77}$$

and $\widehat{A}_{\mathrm{LVE}_\phi^k} = \mathcal{M}_{k-1}^T$ for every $k \geqslant 1$. A result in [5] using said notation is easier to prove in this setting. Indeed, the same reasoning underlying (55) applies to row vector $F^{(k)}$, hence $\frac{\partial}{\partial z_m} F^{(k)} = F^{(k+1)} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k} \right)$, and following Lemma 2.12

$$\dot{\overline{F^{(k)}}} = F^{(k+1)} \sum_{m=1}^{n} \left(\boldsymbol{e}_m \odot \mathrm{Id}_n^{\odot k} \right) \dot{\phi}_m = F^{(k+1)} \left(\dot{\boldsymbol{\phi}} \odot \mathrm{Id}_n^{\odot k} \right) = F^{(k+1)} \left(A_0 \odot \mathrm{Id}_n^{\odot k} \right),$$

implying $\dot{\overline{\left(F^{(k)}\right)^T}} = \left(A_0 \odot \mathrm{Id}_n^{\odot k} \right)^T \left(F^{(k+1)}\right)^T$; placing all terms on one side, and observing Lemma 5.3 and the transpose of expression (76), we obtain:

Proposition 5.4 ([5, Th. 12])**.** *Let F, $\boldsymbol{\phi}$, V be defined as in Lemma 5.3. Then* $\widehat{A}_{\mathrm{LVE}_\phi}^T V = 0$. □

Hence, blocks in $\boldsymbol{V}_1, \left(\boldsymbol{V}_2, \boldsymbol{V}_1\right)^T, \left(\boldsymbol{V}_3, \boldsymbol{V}_2, \boldsymbol{V}_1\right)^T, \ldots$ having all entries in the base field K and satisfying both equations in Proposition 5.4 and 5.3 are candidates for jet blocks $F^{(1)}, F^{(2)}, \ldots$ of a formal first integral. These blocks belonging to the intersection of $\ker \widehat{A}_{\mathrm{LVE}_\phi^k}^T$ and the solution subspace $\mathrm{Sol}_K \left(\mathrm{LVE}_\phi^k \right)^\star$ were called *admissible* solutions of the order-k adjoint system in [5].

This takes us back to the end of Section 3.2. Consider *gauge transformation* ([2, 5, 6, 22]) $\boldsymbol{x} = P\boldsymbol{X}$ transforming linear system $\dot{\xi} = A_1 \xi$ into equivalent

$$\dot{\Xi} = P\left[A_1\right] \Xi := \left(P^{-1} A_1 P - P^{-1} \dot{P} \right) \Xi.$$

Using notation $Y_i = PX_i$, $J_\phi = PX$ and item (e) in Lemma 3.10, we recover the result already seen in previous references, summarised in the extension of gauge transformations to higher dimensions via $P^{\odot k}$:

$$\exp_\odot (X) = \exp_\odot \left(P^{-1} J_\phi\right) = \exp_\odot P^{-1} \exp_\odot J_\phi = \mathrm{diag}\left(\cdots, \left(P^{-1}\right)^{\odot 3}, \left(P^{-1}\right)^{\odot 2}, P^{-1}, 1 \right) \exp_\odot J_\phi,$$

and very simple application of properties seen so far extends the general structure of the gauge transformation to $\Psi = \exp_\odot P^{-1} \exp_\odot J_\phi$:

$$\dot{\Psi} = P\left[A_{\mathrm{LVE}_\phi}\right] \Psi := \left(\exp_\odot P^{-1} A_{\mathrm{LVE}_\phi} \exp_\odot P + \left(\dot{\overline{P^{-1}}} \odot \exp_\odot P^{-1} \right) \exp_\odot P \right) \Psi. \tag{78}$$

The second term $\left(\dot{\overline{P^{-1}}} \odot \exp_\odot P^{-1} \right) \exp_\odot P$ inside the brackets can be simplified into the following:

$$\mathrm{diag}\left(\ldots, k \left[\dot{\overline{P^{-1}}} \odot \left(P^{-1}\right)^{\odot k-1} \right] P^{\odot k}, \ldots, 2 \left(\dot{\overline{P^{-1}}} \odot P^{-1} \right) P^{\odot 2}, -P^{-1}\dot{P}, 0 \right), \tag{79}$$

with $\dot{\overline{P^{-1}}} = -P^{-1}\dot{P}P^{-1}$. The above gauge transformation can be seen as the effect of transformation $\boldsymbol{z} = P\boldsymbol{Z}$ on the jet of (DS). Given a first integral F of the latter, we may always assume $F(\boldsymbol{\phi}) = 0$, which implies $M_F^{1,0} = 0$ and, as seen in (52) or in Lemma 3.6,

$$F_P(\boldsymbol{Z}) = J_F\left(\exp_\odot P\right) \exp_\odot \boldsymbol{Z}.$$

The jet of this formal series is

$$J_{F_P} = J_F\left(\exp_\odot P\right) = \left(\begin{array}{cccc|} \cdots & 0 & 0 & 0 \\ \cdots & F^{(0)}(\boldsymbol{\phi})\, P^{\odot 3} & F^{(2)}(\boldsymbol{\phi})\, P^{\odot 2} & F^{(1)}(\boldsymbol{\phi})\, P \\ \hline \cdots & 0 & 0 & 0 \end{array} \right) \in \mathrm{Mat}^{1,n}(K),$$

and applying (78), Lemmae 5.3 and 5.4, and the identity

$$\left(P^{-1}\right)^{\odot k} \dot{P}^{\odot k} = -\overset{\cdot}{\overline{\left(P^{-1}\right)^{\odot k}}} P^{\odot k},$$

we have just proven the following:

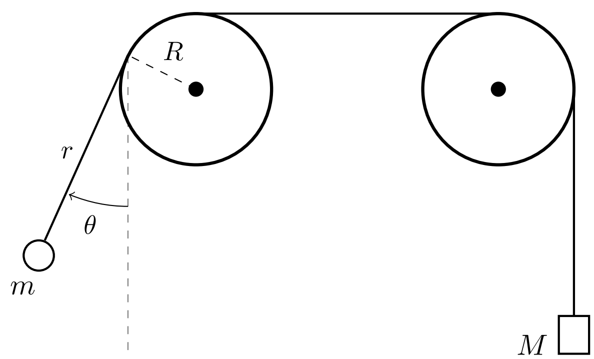
Proposition 5.5. *The transposed jet* $V_P := J^T_{F_P}$ *in the new variables must satisfy*

$$\dot{V}_P = -P\left[A_{\mathrm{LVE}_\phi}\right]^T V_P, \qquad \widehat{A}_{\mathrm{LVE}_\phi}\left(\exp_\odot P^{-1}\right)^T V_P = 0. \qquad \square \tag{80}$$

The key importance in practical examples resides in the choice of the particular solution ϕ and the reduction matrix P, in order to render (80) easier (or more convenient) to solve than its unreduced counterparts, Lemma 5.3 and Proposition 5.4: see also [3, 4].

6 Example

The dynamics of the *Swinging Atwood Machine (SAM)*, summarised in the diagram below,



are governed by Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[\frac{p_1^2}{M_t} + \frac{(p_2 + Rp_1)^2}{mq_1^2}\right] + gq_1(M - m\cos q_2) - gR(Mq_2 - m\sin q_2),$$

where $M_t = M + m + 2I_p/R^2$, I_p is the pulley inertial momentum, $q_1 = r$, $q_2 = \theta$, $p_1 = p_r$, and $p_2 = p_\theta$. We know ([26, Th. 7.5]) the following:

Theorem 6.1. *For every physically consistent value of the parameters, regardless of* I_p *and* R, $\mathcal{H}$ *is meromorphically non-integrable.*

For *SAM without massive pulleys*, i.e. the limit case $I_p = 0$, $R = 0$ and $M_t = M + m$:

$$\mathcal{H}_w = \frac{1}{2}\left(\frac{p_1^2}{M+m} + \frac{p_2^2}{mq_1^2}\right) + gq_1\left(M - m\cos q_2\right),$$

the following holds:

Theorem 6.2.

1. ([15, Theorem 1]) *If* $M > m$ *and*

$$\mu = \frac{M}{m} \neq \mu_p := \frac{p(p+1)}{p(p+1) - 4}$$

for every $p \in \mathbb{Z}, p \geqslant 2$, *then Hamiltonian* $\mathcal{H}_w$ *is non-integrable.*

2. ([29, Equation (16)]) *For $p = 2$, $\mu = \mu_2 = 3$, $X_{\mathcal{H}_w}$ is integrable and has the following first integral:*

$$I = q_1^2 \dot{q}_2 \left(\dot{q}_1 c - \frac{q_1 \dot{q}_2}{2} s \right) + g q_1^2 s c^2 = g q_1^2 c^2 s + p_2 \frac{p_1 q_1 c - 2 p_2 s}{4 m^2 q_1},$$

where $c = \cos(q_2/2)$, $s = \sin(q_2/2)$.

3. ([20, Theorem 4]) *The degenerate cases μ_p, $p \geqslant 2$ in item **1** are non-integrable.* □

Canonical transformation

$$q_1 = Q_1, \quad q_2 = \arccos Q_2, \quad p_1 = P_1, \quad p_2 = -P_2 \sqrt{1 - Q_2^2},$$

performed on $\mathcal{H}_w$ yields Hamiltonian

$$H = g Q_1 (M - m Q_2) + \frac{1}{2} \left(\frac{P_1^2}{M + m} - \frac{P_2^2 \left(Q_2^2 - 1 \right)}{m Q_1^2} \right).$$

Let us apply the techniques introduced in the previous Sections to the variational systems for this Hamiltonian.

6.1 Monodromy matrices and integrability

Consider the particular solution

$$\boldsymbol{\phi} = (Q_1, Q_2, P_1, P_2) = \left(-\frac{g (t-1) t}{2}, -1, -\frac{g (m + M)(2t - 1)}{2}, \frac{g^2 m (t-1) t \left(C_1 - 2 C_1 t + t^2 \right)}{4 (C_1 - t)} \right). \tag{81}$$

In the forthcoming calculations, any value of C_1 different from 0 or 1 will ensure the presence of logarithms in the fundamental matrix, and any value of C_1 different from $1/2$ will avoid division by zero. Choose, for instance, $C_1 = 1/3$. Gauge transformation $\boldsymbol{x} = P \boldsymbol{X}$ with

$$P := \frac{1}{\sqrt{M+m}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{2(M+m)(1-3t)^2}{9gm(t-1)^2 t^2} & 0 & 0 \\ 0 & \frac{(M+m)(1+t)(3t-1)}{9(t-1)^2 t^2} & -1 & 0 \\ -\frac{gm(1+t)}{2(3t-1)} & -\frac{g\left(15m(t-1)t(3t-1)^3 + M(1+t(16+15t(-9+t(37+27(-2+t)t))))\right)}{90(1-3t)^2(t-1)t} & 0 & \frac{9gm(t-1)^2 t^2}{2(1-3t)^2} \end{pmatrix},$$

transforms (VE$_\phi$) into the simplified system $\dot{\boldsymbol{X}} = P[A_1] \boldsymbol{X}$ where

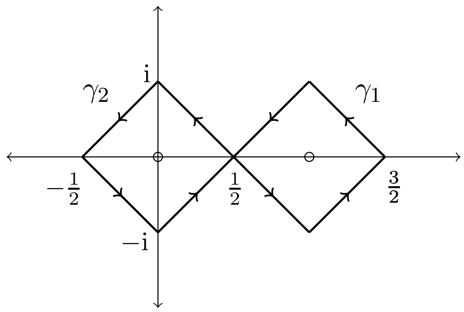
$$P[A_1] = \begin{pmatrix} 0 & -\frac{1}{9}\left(\frac{4}{(t-1)^2} - \frac{1}{t^2} \right) & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{8M(1-3t)}{405m(t-1)^4 t^4} & \frac{1}{9}\left(\frac{4}{(t-1)^2} - \frac{1}{t^2} \right) & 0 \end{pmatrix}. \tag{82}$$

A fundamental matrix for this system is

$$\Psi := e^{\int P[A_1]} = \begin{pmatrix} 1 & \frac{1+3t}{9(t-1)t} & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{4M\left\{ \frac{2}{3}(2t-1)[1+5(t-1)t(6(t-1)t-1)] + t[3+5t(3-2t(11+3t(2t-5)))] - 20(t-1)^3 t^3 \ln \frac{t-1}{t} \right\}}{405m(t-1)^3 t^3} & -\frac{1+3t}{9(t-1)t} & 1 \end{pmatrix},$$

because $P[A_1]$ commutes with its own indefinite integral – an interesting topic for further study is the possible relationship between said commutation and the level of *reduction* of the matrix

in the sense explained in [3], but this is not central to our present work. Our intention is to perform analytical continuation along two different paths, shown below:



Along γ_1, all terms outside of the diagonal in Ψ vanish except for term $(M_{1,\gamma_1})_{4,2}$ which can be obtained from $\dot{\boldsymbol{X}} = P\left[A_1\right]\boldsymbol{X}$ by assuming $X_2 = 1, X_3 = 0$, which yields $\frac{8M(3t-1)}{405m(t-1)^4t^4} + \dot{X}_4 = 0$ and thus the contour integral

$$(M_{1,\gamma_1})_{4,2} = -\int_{\gamma_1} \frac{8M(3t-1)}{405m\,(t-1)^4\,t^4} = \frac{32iM\pi}{81m}.$$

First-order monodromy M_{1,γ_2} along γ_2 is obtained analogously as Id_4 plus sub-diagonal term

$$(M_{1,\gamma_2})_{4,2} = -\int_{\gamma_2} \frac{8M(3t-1)}{405m\,(t-1)^4\,t^4} = -\frac{32iM\pi}{81m}.$$

Second- and third- order monodromies are given by (65) and (68) respectively. The only change therein is replacing A_1, A_2 and A_3 by their gauge transforms defined by (78) and (79) as the lower row block in

$$P\left[A_{\mathrm{LVE}^3_\phi}\right] = \begin{pmatrix} 3\left[Q^{\odot 3}\left(A_1\odot\mathrm{Id}_n^{\odot 2}\right) + \dot{Q}\odot Q^{\odot 2}\right]P^{\odot 3} & & \\ 3Q^{\odot 2}\left(A_2\odot\mathrm{Id}_n\right)P^{\odot 3} & 2\left[Q^{\odot 2}\left(A_1\odot\mathrm{Id}_n\right) + \dot{Q}\odot Q\right]P^{\odot 2} & \\ P^{-1}A_3P^{\odot 3} & P^{-1}A_2P^{\odot 2} & P\left[A_1\right] \end{pmatrix}$$

where $Q := P^{-1}$. Let $Y_1 = \Psi$. Following the order described in (73), we first compute upper indefinite block $Y_1^{\odot 2}$ in the fundamental matrix, as well as the corresponding block $M_{1,\gamma_i}^{\odot 2}$ in the monodromies. We then place indefinite block $Y_1^{\odot 2}$ under the integral sign to obtain both the indefinite integral $Y_1\int Y_1^{-1}\left(P^{-1}A_2P^{\odot 2}\right)\mathrm{Sym}^2\left(Y_1\right)$ defining Y_2 and its contour counterpart defining $Q_{1,2,\gamma_i}$. Using indefinite terms Y_1 and Y_2 (resp. M_{1,γ_i} and $Q_{1,2,\gamma_i}$) we then build terms $Y_1^{\odot 3}$ and $3Y_1\odot Y_2$ (resp. $M_{1,\gamma_i}^{\odot 3}$ and $3M_{1,\gamma_i}\odot Q_{1,2,\gamma_i}$) and we finally plug $Y_1^{\odot 3}$ and $3Y_1\odot Y_2$ into the corresponding contour integral for $Q_{1,3,\gamma_i}$ as in (68) (where A_3 is replaced by $P^{-1}A_3P^{\odot 3}$). In order to decide whether or not we also need indefinite integral Y_3 for higher orders, let us check whether our third-order monodromy matrices commute. We have $C := M_{3,\gamma_1}M_{3,\gamma_2} - M_{3,\gamma_2}M_{3,\gamma_1}$ equal to zero except for the following terms:

$$C_{34,5} = -\frac{\left(\frac{25600}{85293} - \frac{8960}{85293}\mathrm{i}\right)M^2\pi^2}{g^2m^2(M+m)}, \qquad C_{34,11} = \frac{\left(\frac{44400640}{209564901} + \frac{217666158592}{69854967}\mathrm{i}\right)M^3\pi^3}{g^2m^3(M+m)},$$

$$C_{34,13} = \frac{\left(\frac{2220032}{2587221} + \frac{2775040}{7761663}\mathrm{i}\right)M^2\pi^2}{g^2m^2(M+m)}.$$

This, coupled with the fact that all order-k monodromy matrices built from M_{1,γ_i} belong to $\mathrm{Gal}\left(\mathrm{LVE}^k_\phi\right)^\circ$ (since M_{1,γ_i}, being unipotent, belong to $\mathrm{Gal}\left(\mathrm{VE}_\phi\right)^\circ$ and all higher-order fundamental matrices are obtained from quadratures), allows us to complement the non-integrability proof in [20] (see Theorem 6.2, item **3**) using linearised variational equations.

Remark 6.3. Although we used monodromies in our argument, a similar reasoning applies to Galois groups by noting the presence of $\ln(3t-1)$ in Y_3 and $\ln t, \ln(t-1)$ in Y_1, Y_2, Y_3 and using the Picard-Vessiot extension $\mathbb{C}(t)(\ln t, \ln(t-1), \ln(3t-1)) \mid \mathbb{C}(t)(\ln t, \ln(t-1))$ to glean the structure of a generic Galois group matrix on the fundamental matrix Ψ_3 or its pre-gauge counterpart $\operatorname{diag}(P^{\odot 3}, P^{\odot 2}, P)\,\Psi_3$; see e.g. [25].

6.2 Formal first integrals and admissible solutions

Let us now try to apply gauge transforms to the adjoint system for the same Hamiltonian. We have a particular solution

$$\boldsymbol{\phi} = (Q_1, Q_2, P_1, P_2) = \left(-\frac{g}{2}(t-1)\,t, -1, -\frac{g}{2}(M+m)(2t-1), -\frac{g^2 m}{4}(t-1)\,t^2\right),$$

which corresponds to special case $C_1 = 0$ in (81), and gauge transformation $\boldsymbol{x} = P\boldsymbol{X}$ with

$$P := \begin{pmatrix} \frac{1-t}{\sqrt{M+m}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{M+m}}{gm(t-1)^2} & 0 & 0 \\ -\sqrt{M+m} & \frac{\sqrt{M+m}}{(t-1)^2} & \frac{\sqrt{M+m}}{1-t} & 0 \\ \frac{gm(1-t)}{\sqrt{M+m}} & \frac{g(3m-M)}{12(t-1)\sqrt{M+m}} - \frac{gt}{4}\sqrt{M+m} & 0 & \frac{gm(t-1)^2}{\sqrt{M+m}} \end{pmatrix},$$

transforms (VE_ϕ) into the parameter-free, simplified system $\dot{\boldsymbol{X}} = P[A_1]\,\boldsymbol{X}$ where

$$P[A_1] = \begin{pmatrix} 0 & -\frac{1}{(t-1)^3} & \frac{1}{(t-1)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(t-1)^3} & 0 \end{pmatrix}.$$

The principal fundamental matrix for this system is

$$\Psi := \begin{pmatrix} 1 & \frac{1}{2}\left(\frac{1}{(t-1)^2} - 1\right) & \frac{t}{(1-t)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{(t-2)t}{2(t-1)^2} & 1 \end{pmatrix},$$

and $P\left[A_{\mathrm{LVE}_\phi^3}\right]$ is computed as in the previous Subsection. In the following table, the first two columns correspond to order k and total solution space dimension $D_{4,k}$. The latter two display the dimension of the subspace of rational solutions (i.e. those having their entries in the base field $K = \mathbb{C}(t)$), and the subspace of those among the former satisfying (80), respectively:

k	$\dim_{\mathbb{C}} \mathrm{Sol} P\left[\mathrm{LVE}_\phi^k\right]^\star$	$\dim_{\mathbb{C}} \mathrm{Sol}_K P\left[\mathrm{LVE}_\phi^k\right]^\star$	$\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}}\left(P\left[\mathrm{LVE}_\phi^k\right]^\star\right)$
1	4	4	**3**
2	14	14	**9**
3	34	32	**17**

Using solution (81) in Subsection 6.1 and gauge transformation P given in (82), however, $\dim_{\mathbb{C}} \mathrm{Sol}_K P\left[\mathrm{LVE}_\phi^k\right]^\star$ is considerably reduced and we obtain the following table, displaying

smaller bounds on the amount of admissible solutions:

k	$\dim_{\mathbb{C}} \mathrm{Sol} P\left[\mathrm{LVE}_\phi^k\right]^\star$	$\dim_{\mathbb{C}} \mathrm{Sol}_K P\left[\mathrm{LVE}_\phi^k\right]^\star$	$\dim_{\mathbb{C}} \mathrm{Sol}_{\mathrm{adm}}\left(P\left[\mathrm{LVE}_\phi^k\right]^\star\right)$
1	4	3	**2**
2	14	9	**5**
3	34	19	**9**

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